THE HISTORIC
DEVELOPMENT OF LOGIC

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THE PRINCIPLES AND STRUCTURE OF SCIENCE IN THE
CONCEPTION OF MATHEMATICAL THINKERS

BY
FEDERIGO ENRIQUES

Authorized Translation from the Italian by
JEROME ROSENTHAL

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AUTHOR'S PREFACE

The book which Mr. Rosenthal has undertaken to present to the English-reading public is not entirely an erudite history containing everything that has been written and thought about logic. The fact is that whole sections of this discipline—sections that attract the attention of the student—have here been given no place at all. No mention has thus been made of the figures of the syllogism and of the long disquisitions entailed by them.

Our theme is sufficiently well defined by the sub-title of the book. It corresponds to the argument of Aristotle's *Posterior Analytics*. Our aim is here to show the manner in which the principles and structure of rational science were conceived in the course of history. And since mathematics offers the model of such a science, we have naturally been led to deal with the ideas of mathematical thinkers.

We have, however, not limited ourself to a mere exposition of these ideas. We have rather examined them and traced their development in the light of a higher view.

The logical system of the Greeks—their conception of definitions, axioms and postulates—no longer corresponds to the hypothetico-deductive system of the modern mathematical sciences. The ideal of the latter can,

nevertheless, be shown to be contained, to a certain extent, implicitly in the former. The evolution of thought thus presents an almost necessary course with regard to which numerous views and numerous divergences of opinions assume their proper significance.

We have pointed out that the rigorous development of the understanding of the demands of logic, and hence the realization that deduction is entirely relative, finds its expression in the modern critical conception of mathematics. Pure logic, consequently, reveals itself as inadequate to explain the constructive criteria of science. There arises in this way a new epistemological and methodological problem which certain philosophers— from Bacon to John Stuart Mill—tried to solve by opposing to deductive logic a wider inductive logic.
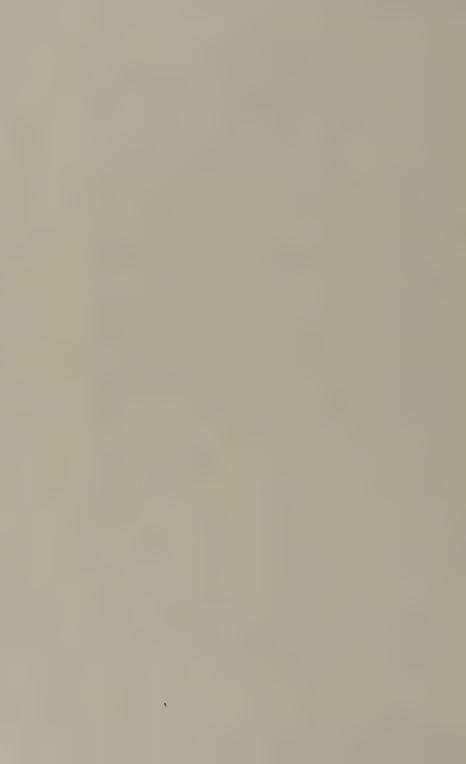
The value of this development and the importance of doing away with the opposition between deduction and induction can be seen clearly in the pages of our book dealing with the origins of experimental rationalism. Deduction is presented here as an instrument of the inductive process of science. But we have given a deeper treatment to our theme in the Appendix, where we have tried to examine in the works of nineteenth century thinkers the development of ideas leading from inductive logic to the actual logic of scientific systems. And, if I am not mistaken, these reflections will have not only an historical interest but also a philosophical, especially in regard to the theory of scientific knowledge.

In addition to what I have said in regard to the constructive criteria of this book I should like to make a few remarks about the English translation. Above all it gives me pleasure to express my gratitude to Mr. Rosen-

thal for the intelligent attention which he has devoted to this work. A translator is generally confronted with a double danger. On the one hand, if he allows himself too much liberty, he runs the risk of not rendering exactly, and even of misrepresenting, the meaning of the author. On the other hand, if the translator clings too faithfully to the expressions of the author he ends by giving the text a cumbersome and obscure form. The latter danger is the most frequent. Mr. Rosenthal, as far as I can judge, has managed to avoid it. He has succeeded in rethinking the text in a form which is most proper to the English language and at the same time in rendering my ideas exactly. Certainly no reader, no more than the author, can have a reason to be displeased with the freedom he has taken in making use of English texts or of English translations of the Greek classics. The bibliographical notes which he has added to our work will prove to be of decided advantage. And I shall be particularly glad if this garment will help it to gain the sympathy of the American public.

<div align="right">FEDERIGO ENRIQUES</div>

Rome, January, 1929.

TRANSLATOR'S PREFACE

The history of logic is certainly a neglected field.
While there are plenty of histories of ethics and psy-
chology, there are no corresponding works, which really
deserve to be called histories, on one of the oldest of
the sciences. The English speaking student finds at his
disposal the historical references in Ueberweg's *Logic* and
A Short History of Logic by Adamson, which originally
appeared as an article in the ninth edition of the
Encyclopædia Britannica. Both are antiquated. The
references in the former are from the nature of the case
fragmentary; besides, they deal mostly with the purely
technical side of logic, with questions relating to the
classification of judgments, moods and figures of the
syllogism, etc. The latter is concerned with the relation
which certain logical doctrines bear to metaphysics and
epistemology rather than with the concrete problems of
logic.

The present work is an important step towards filling
an obvious gap. What above all distinguishes it from the
existing historical works is the freshness of the angle
from which the subject is approached. The center of
interest here is the problem of scientific method, and
especially the structure of the mathematical or deduc-
tive sciences and the significance of the principles upon
which they are based. The book accordingly is largely
devoted to an historical and critical exposition of these

problems. The deductive sciences are dealt with in the first three chapters, while the logic of scientific theories or of the so-called inductive sciences comes up for a detailed discussion in the last chapter.

The author's treatment of his subject is all the more interesting as the clue for the understanding of logical doctrines is shown by him to be found in the works of mathematicians and scientists rather than in the treatises of professional logicians. Thus, in the case of the logic of the Greeks, Aristotle is dethroned from his central position both as the generally accepted creator of logic and as the one who is supposed to have had the greatest influence upon its subsequent development. It is Democritus who is made to come into his own; in this way the wishes of the best thinkers of the Renaissance receive their fulfilment. Similarly, in modern logic the work of Francis Bacon and John Stuart Mill is shown to be of secondary importance in comparison with the critical attempts of those creators of modern science who found it necessary to reflect upon their own procedure.

The present work is short—and it can hardly lay claims to exhaustiveness or completeness. But it has the merit of comprehending the history of logic under a few great developmental lines of thought. Thanks to an imaginative insight combined with great learning both in classical and modern philosophic and scientific literature, Professor Enriques has been able to find similar contents in different forms and to correlate apparently diverse modes of thinking. Suffice it to mention the numerous parallels drawn by him between certain classical and modern tendencies, and especially the highly in-

structive correlation between mathematics and logic
which he traces in the third chapter dealing with the
influence of mathematics on contemporary logic. Within
the brief space allotted to the book the author has man-
aged to crowd in a great deal of invaluable information
bearing on the origin of numerous logical doctrines. The
marginal remarks made by him in the footnotes, in a
seemingly casual way, are as important in this respect
as the focal references given in the body of the text.

No historian can help having a point of view. Pro-
fessor Enriques is no exception. Not that he allows
himself to go beyond the customary boundaries of his-
toric interpretation. He has, however, reserved special
grounds for his views. To such a liberty no one could
object, especially since it gives the reader an excellent
opportunity to come into contact with the ideas of one
of the foremost contemporary thinkers.

The author's point of view is decidedly empiricistic
and psychologistic. He frankly states that logic is for him
a part of psychology. But this does not lead him to the
narrow nominalism or to the static atomistic sensational-
ism of the earlier empiricists.

Professor Enriques qualifies his psychologism as ra-
tional to allow for those ideal tendencies—commonly
called norms—which function in the abstract sciences as
potential limits of what is given in actual experience.
But the author's psychologism means in the case of logic
something more than this. As a renewed and original form
of the Boolean method of basing symbolic logic upon
mental operations, it can be interpreted as an attempt
to bridge the gulf between essence and existence. In
other words, it can be regarded as a means for finding an

existential basis—be it in mental reality—for the prin-
ciples of logic and even for the categories of hypothetico-
deductive systems—things which are usually left to hang
in the air by most rationalists. The scope of this psycholo-
gism is thus wide enough to integrate the operations of
mathematical and symbolic logic with the general activi-
ties of the mind. And the student of the latter discipline
will find particularly interesting the interpretation which
the author gives in this connection of intensional logic
in terms of extensional.

Our author's views of knowledge and science bear
familiar American features; they greatly resemble cer-
tain doctrines of Peirce and Dewey. Knowledge is con-
ceived by him relationally, prospectively, and organically,
and not in terms of discrete primary elements or of imme-
diately given sensations. Similarly, science is not to be
regarded as being based upon absolutely certain princi-
ples leading to conclusions which shine by a borrowed
light. The relation between principles and conclusions in
science is a reciprocal and functional one. In general,
science has to be regarded as dynamic, historic, and
cumulative, as subject to continual criticism and reno-
vation.

Unlike so many other empiricists and positivists Pro-
fessor Enriques believes in the indispensable value of
hypotheses or scientific theories. They are for him, more-
over, not entirely arbitrary and independent of the reality
which they are meant, mysteriously enough, to signify.
His conception of hypotheses as representative serves him
here, too, as a bridge between the concepts of a given
limited science and the general realm of scientific knowl-
edge, and hence as a link with reality at large. It is in

this way that he makes geometry a part of physics, by giving it a concrete interpretation after the manner of Helmholtz. But this does not mean that science is for our author exclusively experimental. Along with the data of experience there function in science also certain rational demands, which manifest themselves, to be sure, as general tendencies only, and not as ready-made, definite, a priori forms imposed upon separately given contents.

If labels mean anything in philosophy, one would have to characterize our author's philosophical views as an interesting variety of critical realism rather than of subjective idealism as might be expected from his empiricism. His whole-hearted adherence to the unlimited possibilities of science as well as his belief in the regulative function of reason entitles one to link him in a certain sense also with rationalism.
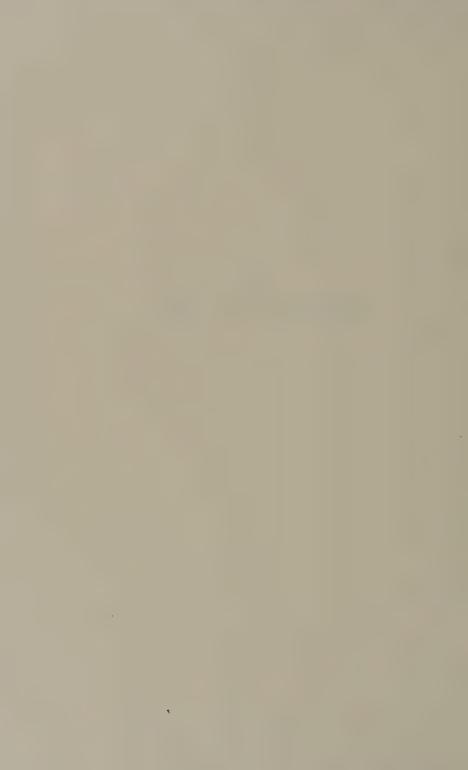
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DEVELOPMENT OF LOGIC

I

THE LOGIC OF THE ANCIENTS

1. Mathematics and the Origin of Logic

"Those who go deep in dialectics," says Aristo of Kos, "may be resembled to crab-eaters; for a mouthful of meat they may spend their time over a heap of shells." W. Hamilton, who quotes this statement,[1] adds a remark which does not seem to have lost its value even to-day: "The modern student loses his time without even a savor of a mouthful."

The young student of mathematics would indeed look in vain to the classical logic into which he was initiated for an adequate conception of the structure of a deductive science like geometry, let alone for an explanation of the value and meaning of the principles of such a discipline. What are axioms, postulates, definitions? What place do they occupy in the edifice of the theory? What criteria determine their choice, and how can we establish whether they are acceptable or not? All these questions will at best be vaguely alluded to in some obscure doctrine dealing with concepts; on the whole they will remain unanswered. Our student will derive little help from the minute syllogistic classifications, although they may enable him to verify what stands in no need of verification, namely the formal coherence of geometrical proofs.

It is important to point out that the mathematician who is concerned with the problem of the structure of

[1] *The Edinburgh Review,* 1833.

his discipline finds himself in regard to logic in the same situation which confronted the thinkers who laid down its foundations. For the development of logic grew precisely out of the critical work of the mathematicians and philosophers who reflected about the nature and structure of mathematical truths.

Aristotle is usually regarded as the father of logic. But whatever original contributions he may have made [1] he can be considered only as a compiler and systematizer of what had been worked out before in this field. We have only to remember the following facts to see the correctness of this statement:

Mathematics had reached quite a high stage of development at the time of Plato; for treatises had begun to be written on its *Elements* since the days of Hippocrates of Kos (about 450 B.C.).

It was exactly at the time of Plato (that is, in the first half of the fourth century B.C.) and in a more or less close connection with the Athenian school of philosophy where Aristotle was trained, that certain mathematical theories were made the object of a profound critical elaboration (Eudoxus and Theaetetus)—an elaboration which is to be regarded as the historical precedent of Euclid's *Elements*.

We have to keep in mind, on the other hand, the extraordinary development which dialectic attained in the discussions of the Sophists. We see this in the case of the

[1] The compliment which Aristotle pays himself at the end of the Sophistical Elenchi for having created a new science seems, as is obvious to any one reading the whole paragraph, to refer to the science of dialectic in the strict sense of the term, and in any case proves nothing against our assertion.

first salaried teachers, of philosophers like Protagoras of Abdera who vindicated the cause of empiricism against the rationalistic metaphysics of the Eleatic School. This shows itself especially among the Megarians and other related thinkers who continued the Eleatic tradition in a formalistic sense in connection with the Socratic circles. The subtlety of some of the sophisms attributed to these thinkers is alone sufficient to show the depth of their analysis. Aristotle often cuts a rather poor figure in his explanations and refutations of these sophisms in the Sophistical Elenchi.

It is noteworthy that Aristotle's polemical writings against unnamed adversaries (for instance, on the necessity and nature of principles in the Posterior Analytics I, 3) show clearly that the logical problem of the structure of deductive science was discussed from various points of view. Some of these views will be seen upon a detailed examination to be nearer to modern conceptions than to those of the Stagirite.

The treatises of Aristotle which were collected under the comprehensive name *Organon* betray a double origin. They have their roots partly in the criticism of mathematics, partly in the practice of discussions. The first two treatises (The Categories and On Interpretation), which form in a certain sense an introduction to the whole work, deal indeed with the classification of isolated words and propositions. The two following ones (The Prior and Posterior Analytics) exhibit logic as a science, as resulting from an analysis of mathematical thinking. The last two (Topics and Sophistical Elenchi) refer to the art of arguing as displayed in the practice of dis-

cussion where we aim not at the true but at the probable. Aristotle reserves for this art the Eleatico-Platonic name "dialectic," while he applies the name "analytic" to the study of the methods of demonstrative science by which we deduce from the possibility of the science the conditions of its structure. (This sense of the term was taken over by Kant in that part of the *Critique of Pure Reason* which constitutes the Transcendental Analytic.) The term "logic" is used by Aristotle for denoting discursive methods which do not start from principles and therefore have no demonstrative value.[1] But this term already occurs in a lost work by Democritus of Abdera (460-360 B.C.): περὶ λογικῶν ἤ κανών [2] and in so far as it is permissible to assume that Aristotle retained its meaning it would reveal a different conception (more relative and formal) of logical reasoning. Such a conception can indeed be met with after Aristotle and especially in the works of the Stoics. Beginning with Zeno of Cition (about 340-265 B.C.) these thinkers designate by the term τὸ λογικόν [3] that part of philosophy which deals with discourse and which includes questions of logic proper as well as those of grammar and rhetoric. It is certainly from Democritus that the contemporary Epicurean school derived, on the other hand, the name "Canonic" by which it denoted rules of method. These remarks tend to show that the influence of the vast work of Aristotle on his successors was not so exclusive as is commonly believed. We shall thus be led to look in the

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[1] This remark was made by Prantl in his *Geschichte der Logik im Abendlande* (Leipzig, 1855), vol. I, pp. 116, 336.

[2] Diels, *Die Fragmente der Vorsokratiker:* Dem. A. 33, B, 10ᵇ.

[3] Diog. Laert., VII, 55 (in Arnim, Diogenes, 16).

works of these successors for traces of older views and especially for those of the master of Abdera.[1]

In order to form an idea about the origin of Greek logic it would be interesting to find out what were the relations, and whether there were any at all, between the criticism of the mathematicians and the subtle disquisitions of the Sophists. Clairaut [2] ascribes the rigor of Euclid's method to the fact that "this geometrician had to convince obstinate Sophists who plumed themselves on their refusal to accept most obvious truths." According to Hoüel, Euclid's dogmatic manner was due to his "desire to shut the mouth of the Sophists, whom Greece, foolishly enough, took all too seriously; hence his habit of always proving that a *thing cannot be* instead of proving that it *is*." [3]

These views have often been contested, for it is difficult to believe that the Sophists had a direct influence upon Euclid or even upon those of his predecessors who had worked out critically the science of mathematics.[4] It is, nevertheless, possible to cite in this connection some references to an antimathematical polemic on the part of

[1] We should like to add that in the opinion of Prantl (op. cit., pp. 515, 561) it was rather the later Peripatetics than the Stoics who introduced the term ἡ λογική as a designation for the science of reasoning, or as a term comprising both logic and rhetoric.

[2] *Élements de géométrie* (Paris, 1741), préf. pp. 10-11.

[3] *Essai critique sur les principes fondamentaux de la géométrie* (Paris, 1867), 1st ed., p. 7.

[4] The friendly relations between Protagoras and the mathematician Theodorus of Cyrene are none the less confirmed by Plato (*Theaetetus,* 161b-162a).

Protagoras [1] and Antiphon [2] who defended the empirical character of geometrical concepts against the rationalistic philosophy. Empiricists commonly make use of arguments of this type, and as far as antiquity is concerned they can all be found in Sextus Empiricus.[3]

But whatever we may think of the views of Clairaut and Hoüel in general, our two authors are certainly wrong in their depreciatory attitude towards the Sophistic movement. There seems to be another and more important link between the logical criticism of the mathematicians and the dialectic of the Sophists, for both have their roots in the Eleatic philosophy. Zeno of Elea is indeed credited by Aristotle with having been the inventor of the *litigious* art called dialectic.[4] On the other hand, the penetrating analysis of Tannery and Zeuthen has brought out clearly the importance and mathematical significance of the famous Zenonian arguments on motion (the flying arrow, Achilles, etc.). The subtle dialectician, in whom tradition saw nothing but a paradoxical quibbler, reveals himself to us as the thinker who paved the way for infinitesimal analysis. It is instructive to realize that it is exactly from considerations of infinitesimals—where thought is exposed to unsuspected fallacies—that the criticism of reasoning takes its origin, thus leading to the discovery of the principle of contradiction and the method of *reductio ad absurdum*.[5] Democritus, who pushed infini-

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[1] Aristotle, *Met.* II, 2 (20).

[2] Cf. Simplicius in Aristotle Phys. (Diels, B. 13).

[3] *Adversus Mathematicos*, I, III.

[4] Cf. Diogenes L., VIII, 57; Sextus, *Adv. Math.*, VII, 6 (in Diels, Zeno, A. 10); Arist. ed. Didot, vol. 5, part 2, p. 92.

[5] Cf. Enriques, "Il procedimento di riduzione all'assurdo," *Bolletino della mathesis*, 1919.

tesimal analysis still further by his discovery of the vol-
ume of a pyramid, is also mentioned by Diogenes Laer-
tius as a continuer of the Zenonian dialectic.

It is necessary to explain here briefly how the origins
of infinitesimal analysis are connected with the criticism
of the principles of geometry and, consequently, with the
development of logic. For the proofs of our statements
we can refer to the works mentioned above [1] as well as
to some of our own writings where this question has been
dealt with in a more detailed manner.[2]

According to what Proclus tells us in his commentary
to the first book of Euclid, the main geometrical theories
which make up the *Elements* were worked out and dem-
onstrated by the Pythagoreans. In the opinion of Zeu-
then, the starting point for this movement was the at-
tempt to establish the relation between the squares on
the hypotenuse and on the legs of a right triangle, known
as the Pythagorean theorem. On the other hand, there
are numerous reasons for assuming that the Pythagorean
geometry was based on a theory of proportions or meas-
ure, which was derived from the empirical concept of
extended points considered as the unitary elements of
all things (monads). The Pythagorean saying that "things
are numbers" has to be interpreted as meaning that
bodies or geometrical figures, which are conceived at this
stage of development concretely, are aggregates of points,
that is, units having position.

But the monadic hypothesis involves the commensura-
bility of any two segments, a fact which makes measure-

[1] Cf. especially P. Tannery, *Pour la science hellène*, chap. X.

[2] "La polemica eleatica per il concetto razionale della geometria,"
Periodico di matematiche, March, 1923.

ment possible. Such a conception could not fail to clash —in the Pythagorean school itself—with the discovery that the diagonal and the side of a square are incommensurable. While the Pythagoreans were wrestling with this difficulty, other philosophers, who also came from the same circles,[1] embarked upon a criticism of the concepts of geometry. They came to the conclusion that rational thought can avoid contradiction only by regarding points as devoid of extension, lines as length without breadth, and surfaces without thickness. In this way they were naturally led to the first considerations of infinitesimals. These rationalistic critics are the Eleatics—Parmenides and his disciple Zeno. Their speculation, which belongs to the first half of the fifth century B.C., marks a decisive point in the history of Greek philosophy, for it distinctly proclaims the rights of reason for the first time. Coherent thought becomes the criterion of *truth*, that is to say, of metaphysical existence in contradistinction to *probable opinion*, which deals with sense reality.

It was this rationalism and the consequent transference of faith from phenomenal appearances to the principles of reason that gave birth—as we have said—to the dialectical method, which is the germ of logic. This was to mature in the storm of controversies between empiricists and rationalists, and through the efforts of the latter infinitesimal analysis was to be developed further (Democritus) and its principles were to be worked out critically (Eudoxus).

But since this criticism struck at the roots of the

[1] Parmenides is counted among the Pythagoreans in the catalogue of Iamblichus (Diels, Pyth, 45, A.), and Diogenes Laertius speaks of his relations to other Pythagoreans.

theory of proportions and incommensurables, it involved
the whole problem of the rigorous structure of geometry.
Logical investigation could thus not confine itself any
longer to an analysis of the subtle methods of deduction;
it had naturally to be extended so as to embrace the struc-
ture of the science and the evaluation of its principles.

2. THE VIEWS OF PLATO ON THE LOGICAL STRUCTURE OF MATHEMATICAL SCIENCE

The views of Plato are very interesting in connection
with the questions we have just discussed. To be sure,
the influence which the Athenian philosopher could have
exercised on mathematical thinkers like Eudoxus and
Theaetetus has been somewhat exaggerated by Zeuthen,[1]
who characterizes the critical movement of that period
as "the Platonic reform of mathematics."

Let us quote a few passages from Plato:

The Republic (510, c, d, e) "Students of geometry
and arithmetic assume in their demonstrations the odd,
and the even, and figures, and three kinds of angles
and other similar suppositions. These assumptions are
taken as a basis for demonstration, as if they were evi-
dent and as if the knowledge of them were certain. These
students, therefore, do not deign to give any account of
them either to themselves or to others. They rather pro-
ceed from these to further conclusions, arriving at last
at what they intended to demonstrate. . . . Although
they use for this purpose visible figures and reason about

[1] "Sur la réforme qu'a subie la mathématique de Platon à Euclide et
grâce à laquelle elle est devenue science raisonée" (in Danish with a
French summary: Transactions of the Copenhagen Academy, 1917).

them, they are not thinking of the figures but of the entities of which the figures used are images; they think of the square in itself and of the diagonal in itself rather than of the square and the diagonal which they draw. And in this sense they use all the figures which they draw or make (in a certain sense like shadows and images reflected from the water) as representations, seeking to get through them at their originals which are visible to the understanding (διάνοια) alone. . . .

"Of this kind I spoke as the intelligible, and I meant to say that in the search after it the soul is compelled to use assumptions and does not go back to the beginning, because it is unable to rise above its assumptions, but employing the sense objects of this world as images, for the sense objects are thought to have a greater distinctness in relation to the assumptions. On the other hand, reason uses the power of dialectic and regards the assumptions not as first principles but only as assumptions, as starting points, arriving at what has no longer any assumptions, that is, at a universal principle, and having attained it, it clings to the consequences springing from it; in this way it arrives at the end without the help of any sense object, that is, it proceeds through ideas from idea to idea, ending in ideas." Hence the distinction made between the reason of the dialectician (νοῦς, νόησις) and the understanding of the geometrician (διάνοια) as "lying midway between opinion and reason."

The same distinction occurs in another place: Rep. (533, c, ...) "Geometry and the related sciences dream about being, but they cannot behold it with open eyes as long as they use postulates and cling to them. How is it possible to call science a discipline which

is ignorant about its beginning and whose middle and end rest on something which it does not know?"

It is not quite easy to understand these views. We must reject, in the first place, the most common interpretation which sees a fundamental difference between the reason of the dialectician and the understanding of the geometrician. For it is impossible to attach any meaning to the Platonic ideas, if we do not regard them as having the same kind of "existence" as the one predicated of mathematical forms and relations in nature.

The apparent contradiction between this manner of interpreting the doctrine and the words of the text quoted above disappears as soon as we realize that the lower status assigned to mathematics in comparison with dialectic refers not so much to pure mathematics, which can be constructed as a science ($\mu\alpha\theta\eta\mu\alpha\tau\alpha$) in accordance with the ideal of our philosopher, as to mathematics considered as an art ($\tau\epsilon\chi\nu\alpha\iota$).[1] In support of this contention we can quote other passages from the same dialogue, for instance:

"Even those who have little acquaintance with geometry will not contest that the nature of this science is in flat contradiction with the language used by those who practise it. Their terminology is too narrow and ridiculous, for they are always speaking of squaring, extending, applying, confusing thus the necessities of geometry with those of daily life; whereas pure knowledge is the real object of the whole science."

But what is the logical structure of the ideal geometry

[1] Cf. G. Milhaud, *Les philosophes géomètres de la Grèce* (Paris, 1900), (ch. II), and Enriques, *Scienza e razionalismo* (Bologna, 1912), p. 50.

cherished by Plato? Upon what basis does he want to erect its principles?

The passages quoted above show clearly enough what our philosopher would eliminate in order to bestow the character of rationality upon this science. He would, namely, do away with those demands which are laid at the basis of demonstration under the name of postulates (αἰτήματα) and thanks to which certain constructions are assumed as possible through the performance of practical operations on sense models. The basis of a geometry constructed in conformity with the criteria of dialectic would thus consist of pure definitions (the purpose of dialectic is exactly to define concepts) or of evident principles, such as axioms, which would be for Plato inborn ideas, according to the doctrine of recollection expounded in *Meno*. Pure reason (νοῦς) appears in this way as the source of the elementary geometrical properties in the apprehension of which the understanding (διάνοια) makes visible figures play the part of occasions.

3. THE CONCEPT OF DEMONSTRATIVE SCIENCE IN ARISTOTLE: POSTERIOR ANALYTICS

When we turn now to Aristotle's Analytics, we find there more precise information in regard to the criteria adopted by geometricians in the logical construction of their science. It would be interesting to compare these criteria with those actually functioning in Euclid's *Elements*.

Already at the beginning of the Prior Analytics our author defines in the following words the objects of the science which he is going to study: "It is first requisite to

establish what is the subject and what is the purpose of the present treatise; the subject is demonstration and the purpose is demonstrative science." (ἐπιστήμη ἀποδεικτική.) He then lays down, in the same work, the doctrine of the syllogism and goes on (in the Posterior Analytics) to examine the logical structure of deductive sciences, constantly referring to mathematics.

The latter work, to which we shall pay special attention,[1] begins with the following statement: "All knowledge, whether taught or acquired, always arises from pre-existing knowledge. Observation shows that this is true of all the sciences; this is indeed the procedure of mathematics and, without exception, of all other arts. . . . It follows necessarily from the very concept of knowledge that demonstrative science starts from true and immediate principles, from principles which are better known than the conclusions of which they are the cause and to which they are prior."[2] Aristotle examines (ibid. I, 3) and rejects objections made by two kinds of adversaries asserting that (1) either there are no principles and demonstration is, consequently, impossible, since it leads to an infinite regress; (2) or, in case there are principles, the method of demonstration is wholly relative so that the principles could be proven by the conclusions just as the conclusions could be proven by the principles, a thing involving us in a vicious circle.

It would be interesting to know who these adversaries were. The first objection belonged perhaps to the anti-mathematical polemic of the empiricistic philosophers,

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[1] Cf. Enriques, "Il concetto della logica dimostrativa secondo Aristotele" in *Rivista di filosofia*, January, 1918.

[2] An. Post. I, 2 (6).

while the second may have emanated from Megarian circles which were permeated by the Eleatic relativism, or from Democritus, or other mathematical critics of the principles of science. In any case, the view expressed here—which is only apparently illogical—offers a striking similarity to certain modern tendencies, as we shall have a chance to see later.

Aristotle had to combat this relativism. His whole metaphysics, which was inspired by the Platonic doctrine of ideas and which was made the basis of his logic, exactly represents a reaction against the relativistic tendencies of contemporary philosophy. In this relativism, which had found its way from pre-Socratic science into the field of customs and religious beliefs, he saw a menace for the whole framework of Greek social life. The parallelism which the Eleatics had discovered between being and thought and which they had interpreted as a projection into reality of the arbitrariness characteristic of free criticism assumes an inverse meaning in the Socratic-Platonic doctrine. The ontological doctrine of ideas rests indeed on the presupposition that there is an absolute order of truths, confronting thought as data on the basis of which it has to shape the structure of logic, its own science. Plato thus sees in the classification of geometrical forms an exemplification of the hierarchy of natural species which comes to light in the general method of definition and division characteristic of dialectic. And similarly for Aristotle, the necessary and irreversible connection of cause and effect displayed by nature finds its reflection in the relation of premises to conclusions in demonstrative science. It is for this reason that demonstrative science possesses an irreversible nat-

ural order; hence the absolutely *indemonstrable* character
of its principles:

"The principles from which we start must be inde-
monstrable, otherwise a man will not know them, be-
cause he does not possess the demonstration of them, for
to know in a non-accidental fashion things which can be
demonstrated means to possess their demonstration."
Posterior Analytics I, 2 (9).

Upon a further examination of the Posterior Analytics
we learn that the principles of science can be distin-
guished into several kinds:

1) Terms or definitions (ὄροι), that is, assumptions
about the meaning of words (in modern terminology:
assumptions about primitive non-defined concepts) and
definitions proper.[1]

2) Assumptions about the existence of genera and
their species, that is, of the things denoted by the terms.

3) Immediately evident propositions which we must
necessarily know in order to understand anything; these
are called axioms (ἀξιώματα) "because there are propo-
sitions of this nature, and it is to them that this term is
commonly applied."[2]

4) Finally also hypotheses or postulates (αἰτήματα),
which are actually used in the teaching of mathematics
(or even in discussions), where the learner is expected

[1] The logical theory of definition is treated by Aristotle in the Pos-
terior Analytics, especially in chapters 9, 12, and 13. The rule is pre-
scribed there to restrict successively the extension of the genus, by
adding, in a natural order, the delimiting differentiae, until they com-
pletely circumscribe the extension of the subject to be defined.

[2] This term is used by the Pythagoreans according to Iamblichus (in
Diels. D. 6).

to grant the existence of things of which he either has no idea at all or of which he has a contrary idea.

Aristotle's views seem here to be somewhat obscure. On the one hand, he appears to assume with Plato that postulates can be eliminated: "A postulate is what is laid down without proof, although it can be proven, and what we use without demonstration" (I, 10, 8). On the other hand, we are told (9)—evidently in reference to the ideas of the geometricians—that "definitions are not hypotheses, because they do not state whether the things defined exist or not . . ." He probably thought that knowledge has to be based only on existential suppositions having a necessary character, since they are true by themselves (καθ'αὑτά). These "cannot be regarded as hypotheses or postulates . . ." (I, 10, 7), for "demonstration refers not to external speech, but to the speech of the soul." Our philosopher appeals here to that feeling of the self-evident nature of thought which Plato characterizes in the *Theaetetus,* almost in the same words, as inner sincerity.[1]

Aristotle, nevertheless, criticizes the Platonic doctrine of recollection and denies that there are innate principles. Universal knowledge of principles is, according to him, undoubtedly acquired through sensation. What makes it possible for us to have such knowledge is, the unity of experience existing in the soul. For despite the multiplicity of objects, the soul possesses the faculty for apprehending the similar and identical elements found in particulars and for recognizing them as thought data.

[1] "Thought is a conversation which the soul holds with itself about the things which it examines." (189 c.)

"Not even in sleep did you ever venture to say to yourself that odd is even or anything of the kind" (190).

(Posterior Analytics II, 15 (5, 6, 7).) This in no way impairs the absolute truth which the understanding (διάνοια), the foundation of science, bestows upon its principles. (II, 1-5 (8).)

4. THE PRINCIPLES IN EUCLID'S ELEMENTS

It would be helpful to compare the doctrines of Aristotle with those revealed by the structure of Euclid's *Elements*.[1] We find here three kinds of principles:

1) terms or definitions (ὅροι),
2) postulates (αἰτήματα),
3) common notions (κοιναὶ ἔννοιαι).

We cannot enter here upon a thorough analysis of these concepts, which, to be sure, are far from satisfactory. Tannery has even contested their authenticity. We shall confine ourselves to a few logical remarks, following the criticism made by Zeuthen.[2]

We shall first consider a question of terminology.

Many students wonder why Euclid uses the term "common notions" for denoting what Aristotle, together with the Pythagoreans, calls "axioms." This is all the more peculiar as the word ἔννοια is believed to have appeared only much later in the language of the Stoics. Now it is important to point out that the same term

[1] Heiberg, *Euclidis opera omnia* (Leipzig, 1883-88), B. G. Teubner. According to the commentator Proclus of Byzantium (412-485 A.D.), Euclid lived at the time of Ptolemy in Alexandria. We may thus infer that the *Elements* were written about 300 B.C. (The works of Aristotle known to us seem to belong to the last decade of his life, which ended in 322 B.C.)

[2] Cf. his *Histoire des mathématiques,* translated from the Danish by Mascart (Paris, 1902): no. 14, 69, 94.

occurs in Democritus,[1] and with a meaning which we shall consider later. This remark gains special interest through the fact that Democritus is also known to have written *Elements* about a hundred years before Euclid. This work is not mentioned in the historical summary of Proclus. Thrasyllus has, however, transmitted to us its titles,[2] which suggest an arrangement of subject matter similar to the one adopted by Euclid. It is thus permissible to assume that axioms were called by Democritus "notions" or "common notions," and that Euclid retained the terminology of his famous predecessor when he undertook to give this subject a systematic form in conformity with the critical progress of the age.[3]

We may add that the difference between common notions or axioms, and postulates is explained by Geminus in Proclus as being similar to the one existing between theorems and problems, or between identities and equations.[4] Axioms and theorems express relations which establish certain properties as following from other given properties. Postulates and problems, on the other hand, prescribe the performance of elementary constructions, a thing which was for the Greeks equivalent to the assertion of the *existence* of particular entities upon which certain conditions are imposed. This constructive character seems to be absent only from the fourth postulate

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[1] Cf. Sextus (in Diels, A. 111).

[2] Γεωμετρικῶν (A,B?) Ἀριθμοί, Περι ἀλόγων γραμμῶν καί ναστῶν Ᾱ,Β̄. (Cf. Deils, B. 11ⁿ, 11°, 11ᵖ).

[3] In support of our view we can perhaps refer to a passage of the well-known commentary *Procli Diadochi in primum Euclidis Elementorum librum commentarii* (ed. Friedlein, p. 194, lines 8-9), in which Proclus seems to allude to the habit of geometricians of calling "common notions" what Aristotle calls "axioms."

[4] Loc. cit. and ff. Cf. Vailati, *Scritti*, p. 547.

(all right angles are equal). But Zeuthen explains this statement as being complementary to the second postulate, which asserts that the prolongation of a straight line is univocally determined.

Proclus finds also another difference between axioms and postulates. The latter are principles special to geometry; the former are common to the various sciences. In reality it is a question here of the general properties of equality and inequality displayed by magnitudes.

Finally the difference between two kinds of principles is also in harmony with the criterion of Aristotle, who sees in axioms necessary and indemonstrable truths, because they are self-evident ($\kappa \alpha \theta$' $\dot{\epsilon} \alpha \upsilon \tau \dot{\alpha}$), whereas he regards postulates as truths which participate in another kind of evidence (sense evidence) and which do not follow with equal necessity ($\dot{\epsilon} \xi$ $\dot{\alpha} \nu \dot{\alpha} \gamma \kappa \eta s$) from the meaning of the terms appearing in them. The nature of the principles which are stated by Euclid as common notions seems indeed to satisfy this criterion.

But if some geometricians, according to Proclus, refused to distinguish between axioms and postulates, we have no evidence to show that they also rejected the meaning which Aristotle and probably also others (following the metaphysics of common sense) attached to this distinction. This is done by modern criticism, which exactly for this reason sees in the primitive propositions of science postulates that have to be assumed in any deductive theory as data preceding the development of the theory itself.

Some light is thrown on these questions by a statement made by Proclus (*loc. cit.* p. 194) in regard to the attempt ascribed to Apollonius at demonstrating the first axiom

(things equal to another thing are equal to each other).
This attempt is briefly summarized as follows: "Let A be
equal to B, and the latter to C; I say that A is also equal
to C. For, since A is equal to B, it occupies the same space
(τόπος) with it; and since B is equal to C it occupies the
same space with it." This mode of reasoning suggests
perhaps that Apollonius resorts here to ideal experiments
with motion in order to reduce the Euclidean concept of
geometrical equality to the case of the superposition of
figures. In this way he was made to assume erroneously
that the transitive property of the first axiom could be
regarded as an instance of a pure identical proposition.
But it is exactly the appeal to such experiments which
shows (Helmholtz and Stoltz) that the first axiom has
a synthetic significance and that it cannot be consid-
ered as an analytic proposition, true by definition. But
whatever the case may be, our quotation entitles us to
assume that after Euclid the criticism of principles was
considerably furthered by Apollonius, thanks to the sharp-
ness of analysis with which we gladly credit the great
geometrician of Perga.

We return to Euclid in order to examine briefly the
principles which he denoted by the name of ὅροι: terms or
definitions. If these are to be regarded as definitions, we
cannot help pointing out their weakness; for they often
are nothing but descriptions indicative of the psychologi-
cal genesis of concepts. This is, for instance, the case
with definitions 3 and 5, where we read that the extremi-
ties of a line are points and that the extremities of a sur-
face are lines. But these and other similar explanations

are probably to be taken in connection with the preced-
ing historical tradition, as reminiscent of those char-
acters through which the entities of rational geometry
are made to appear as idealizations of sense experience.
Thus the definitions 1, 2, and 5 remind us of the fact
that, in accordance with the results of the Eleatic
criticism, the point is without extension, the line is
breadthless length, and the surface is without thickness.[1]

Even those principles which present themselves as
definitions proper do not always satisfy the fundamental
Aristotelian criterion according to which the totality of
attributes in a definition restricts the genus so that the
notion defined cannot belong to any concept having a
wider extension. It is for this reason that the fourth defini-
tion, "A straight line is a line which lies evenly with the
points on itself," seems to be insufficient. For if this is
interpreted in the customary fashion, "A straight line is
a line which can be divided by any of its points into two
equal parts," we get a property which is not characteris-
tic of a straight line, as it belongs also to the helix (Cf.
Apollonius in Proclus: 105, 5).

We have to add that Euclid does not confine himself
to the assumption of the existence of what is immedi-
ately denoted by some of his terms. He also seems to
smuggle in surreptitiously, by means of definitions,
existential hypotheses in places where, by analogy with
the criteria adopted in other cases, we have good reasons
to expect the explicit statement of a postulate. This hap-
pens especially in the case of the intersection of straight
lines and circles; the assumptions employed in proposi-
tions 1, 12, and 22 seem to find their justification, accord-

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[1] Cf. Proclus loc. cit., p. 94, line 11.

ing to Zeuthen, in definition 15 where the circle is described as a plane figure *contained by one line*. But it is unnecessary to dwell upon such shortcomings, for they affect only the technical side of the matter and hardly its essential logical tendencies. Remaining within the Euclidean world of ideas, we shall only have to complete the postulates by explicitly asserting the cases of the actual existence of points of intersection of circle with straight line and of circle with circle, as they occur in elementary constructions. It is interesting to point out that these existential hypotheses, which ancient geometry introduced in single instances by means of appropriate constructions, can be nowadays deduced from the sole principle of continuity.[1] The assertion of existence becomes in this way independent of the search for constructive devices, which often tend to grow quite complicated. This marks a progress in conformity with the tendency advocated by Plato, who, as we have seen, had an aversion for whatever was practical or mechanical in the formulation of postulates.

Note. In connection with what we have said about the geometry of Euclid we should like to add that Archimedes [2] seems to classify and distinguish mathematical principles in a different manner. In a letter to Dositheus he calls "axioms" ($\dot{\alpha}\xi\iota\dot{\omega}\mu\alpha\tau\alpha$) definitions accompanied by existential suppositions: for instance, there are in a plane certain terminated bent lines which lie wholly on the same

[1] Cf. article 5 by G. Vitali in *Questioni rigardanti le matematiche elementari,* collected and arranged by F. Enriques (Bologna, 1912), vol. I.

[2] "De sphera et cylindro" in *Archimedis opera omnia cum commentariis Eutocii,* ed. Heiberg (Leipzig, 1910). Cf. *The Works of Archimedes,* ed. by Heath (Cambridge, 1897).

side, etc.; these are called concave. On the other hand, he
applies the name "assumptions" ($\lambda\alpha\mu\beta\alpha\nu\acute{o}\mu\nu\epsilon\nu\alpha$) to certain
principles (extremely elegant postulates or previously
established propositions) from which his work starts.
Eutocius in his commentary replaces the Archimedean
$\dot{\alpha}\xi\iota\acute{\omega}\mu\alpha\tau\alpha$ by the term $\acute{o}\rho\omicron\iota$.

5. SYNTHETIC CONSIDERATIONS ON THE LOGIC OF THE GREEKS

If we summarize our impressions of the logic of the
Greeks, especially keeping in mind Aristotle's Posterior
Analytics and Euclid's *Elements*, and ask ourselves to
what extent their criteria are acceptable or complete, we
shall arrive at the following results:

1) Greek logic rests on a naïve realism which regards
thought as a copy or direct envisagement of the external
world.   The Pythagorean "number," the Eleatic "con-
tinuous space," are thus conceived concretely, as an
imitation of that cosmic substance which is imagined to
constitute the natural substratum ($\phi\acute{v}\sigma\iota s$) of all things.
The realistic assumption finds its typical expression in
the Platonic doctrine of ideas, which forms the meta-
physical basis for Aristotle's logic. It is this realism that
is responsible for the necessary character of principles
and hence for the belief in a natural order of science
based upon absolutely indemonstrable premises. This
belief receives at least a partial correction in the doc-
trines of the Geometricians.

2) This realism is also the source of the cardinal weak-
ness of the theory of definition. The obscurities of
Aristotle's treatise and the imperfections of Euclid, in
general, the errors of criticism occurring in these works,

can all be traced back to this presupposition as a common root.

It is believed that words correspond to entities of a transcendent intelligible world, and that their meaning can be fixed univocally. Hence the demand that in logical deduction we must not only state explicitly our premises as axioms or postulates but we have also to ascertain the *meaning* of the terms of our judgments, since they have to reflect that reality (geometrical, etc.) which forms the object of thought. But this means that in the process of thinking we are allowed to make tacit *appeals to intuition,* which have only to be made explicit in order to become translated into new axioms. But if intuition (or vision of the meaning) is always presupposed in our rational procedures, how can we ever be sure that the axioms form a complete system? Strictly speaking it is even impossible to define the meaning of such a question. And therefore it is difficult to understand why the necessity is felt for introducing some axioms rather than others, although they are all declared to be evident, necessary, etc.

3) We have to add that the Aristotelian analysis of reasoning which starts from the theory of the syllogism (Prior Analytics) also has its roots in the metaphysical presuppositions of logic. It is especially connected with the fact that the Greeks in general conceived the intelligible reality represented in science after the static pattern furnished by the classification of geometric forms. This is indeed the case of the Eleatic ontology which left its mark upon the Platonic doctrine; Aristotle really never transcended the latter.[1] It was Democritus only

[1] Cf. Ch. Werner, *Aristote et l'idéalisme Platonicien* (Paris, 1910).

who, as we shall see later, rose to the conception of a rational science of motion. His philosophical views, however, were to find an adequate development only 2000 years later at the time of the Renaissance.

We must point out here that certain shortcomings in the Aristotelian analysis of reasoning have been overlooked in the objections raised against the theory of the syllogism by the English empiricists (Bacon and John Stuart Mill), who opposed to deduction an inductive method derived from the generalizations of experience. This becomes especially obvious when reasoning is taken in its strict forms, which, according to the view of the Greek philosopher, alone belonged to demonstrative logic proper. The brief remarks which Aristotle devotes to complete induction in the Prior Analytics certainly do not make up for the lack of an analysis of those constructive logical operations (the meaning of particles like "and," "or," etc.) which function along with the syllogism in the development of mathematical proofs. This gap makes itself felt in the theory of definition, which precisely brings to light the constructive work of thought.

4) Finally we have to notice that this realism manifests itself also in a naïve conception of language. Greek philosophy—whether it believed with Plato in the *Cratylus* in the natural origin of language, or whether it laid stress with Democritus [1] and Aristotle [2] on the con-

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[1] Proclus in his commentary on the *Cratylus* mentions this opinion of Democritus, which is based on the homonymy and synonymy of words, on the change of terms, and on the lack of analogy in the formation of certain verbal expressions. (Cf. the notes of Cousin to the French translation of the *Cratylus*.)

[2] On Interpretation, 2 (I).

ventional elements of language—did not succeed in detecting the essential variety of language. It did not realize that things can be represented in different ways, a fact which brings to light the free activity of the speaker and which also explains why perfect translation is impossible.

Aristotle says indeed: "Spoken words are the images of the ideas called forth in the soul and written words are a copy of spoken language. Just as writing is not identical for all men, so speaking differs among men. But the ideas of the soul of which words are immediate signs are the same for all men, just as the things which these ideas exactly represent are identical for every one."— On Interpretation I, 1.

It is clear how such a doctrine accounts for that confusion between linguistic and logical analysis which culminates in the Aristotelian attempt at deriving a classification of "categories" from grammatical forms.

6. The Logic of Democritus and Its Influence upon the Stoics and Epicureans

In the preceding pages we have tried to study Greek thought as it is reflected in the scientific systems which have reached us. But in order to understand the further development of Greek logic in the post-Aristotelian schools of philosophy, it is necessary to take into account the influence which the predecessors of the Stagirite seem to have exercised upon the movement of ideas.

This movement can be described in the main as tending to emancipate thought from ontologism. Of course, in so far as this mode of thinking expresses the metaphysics of common sense it survives the Platonic-Aristotelian

ideology. The emancipatory tendency shows itself in two ways:

1) In a progress towards logical formalism, which starts from the forms of discourse studied in the Prior Analytics. We meet with this progress already in the first Peripatetics, in men like Eudemus, the author of a history of mathematics, and Theophrastus, the collector of the opinions of the cosmologists, and especially in the Stoics, those heirs of the Megarian dialecticians;

2) In a revision of the principles of the theory of knowledge having for its object the origin and significance of the general concepts from which demonstrative science starts. We are here confronted with views which have to be traced back to the great predecessors of Plato and Aristotle. Owing to the importance of this fact we shall dwell a little longer upon these thinkers.

When we try now to reconstruct inductively the ideas of these predecessors, the figure of Democritus looms above all others. Democritus, who lived 100 years and who was born in 460 B.C.—forty years after Anaxagoras and twenty-five years after his fellow townsman Protagoras, who is the greatest representative of the sophistic school—, has to be considered as an elder contemporary of Plato (427-348/7). It is only the prejudices which dominated the reconstruction of Greek thought in the nineteenth century that stood in the way of a closer study of the relations between the two philosophers. Thus Democritus was relegated, in open disregard of chronology, to the pre-Socratics and even to the pre-Sophists.[1]

[1] Windelband and Burnet form an exception; they place the Abderite in the right chronological order, but they do not do him justice in proportion to the importance of his scientific work.

Democritus is the great founder of the atomic theory, although he was preceded in this by Leucippus. Atomism was developed by him as a kinetic cosmological doctrine by means of which he arrived at a rigorous mechanistic determinism, and probably also at the discovery of principles (mass, inertia) which Galilei reconstructed 2000 years later, resuming the fundamental intuitions of his remote predecessor. On account of his rigid mechanistic view which excludes every form of teleology, Democritus is usually regarded as the father of materialism. And this is exactly the source of the prejudice from which history in the nineteenth century—influenced as its development was especially by the Hegelian philosophy—could never emancipate itself entirely. A closer examination, however, would have revealed in Democritus also the father of spiritualism (just as Leibniz seems to have intuited), and it might perhaps also have traced back to him the attempt to prove the immortality of the soul by its "simplicity" or "indivisibility," as stated in the *Phaedo* 78, *b, c*.[1]

The works of Democritus, the titles of which have been transmitted to us by Thrasyllus, form an imposing array. They refer to the most various subjects, such as mathematics, physics, biology, agriculture, poetics, grammar, the theory of knowledge, etc. Among the most beautiful fragments are the ethical ones, which have been preserved by Stobaeus.

About the philosophy of Democritus and especially about his doctrine of knowledge we are informed by Sextus Empiricus, who characterizes the Abderite together

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[1] I intend to prove this elsewhere through a comparison of the Aristotelian texts.

with Plato as upholders of the truth of thought objects
(τὰ νοητά) in contradistinction to Protagoras.[1] We have
thus to do here with a rationalism which is in opposition
to the Protagorean empiricism. But since the empiricism
of the Sophists had originated in its turn as a reaction of
a positivistic nature against the rationalistic metaphysics
of the Eleatic school, it is natural that Democritus had
to take into account the fundamental demand which the
Sophists had put forward. He could not simply take as an
object of science a Truth (ἀλήθεια) indifferent to opinion
(δόξα) which deals with sense objects. He had instead to
look for a rationalization of the empirically given, that
is, for a truth apt to save appearances (σῶζειν τὰ
φαινόμενα). Expressed in the technical language of the
time this meant that the task of science was to represent
true opinion or opinion rendered true by thought. It is
this Democritean conception of science as δόξα ἀληθὴς
μετὰ λόγο which is referred to and discussed in Plato's
Theaetetus, and an analytic comparison of this text with
other texts in Plato and Aristotle shows that this doctrine
has to be ascribed to Democritus.[2]

But since the rational explanation of phenomena pre-
supposes *concepts* by means of which the things of the
empirical world are represented in a unified manner, it is
interesting to know upon what Democritus based their
presence in the human mind. We have fortunately some
indications in regard to this point.

1) We are told by Aristotle that Democritus was the
first to deal with definitions of physical objects, and

[1] Diels, A. 59; cf. A. 114.
[2] Cf. F. Enriques, "La teoria democritea della scienza nel dialoghi di
Platone," *Rivista di Filosofia,* 1920, no. 1.

that with Socrates, on the other hand, the employment
of definition grew and became especially extended to
moral notions.[1] This suggests that Democritus initiated
that manner of defining which is characteristic of the
Socratic school and in which our task is to discover the
common properties of the things corresponding to the
notion defined. It is more difficult to establish whether
Democritus, like Socrates, also appealed to the common
notions which all human beings are supposed to form in
regard to given objects. This criterion may, however, have
been derived from Heraclitus to whom Socrates himself
seems to have been indebted.

2) In a fragment of the above mentioned logical
work of Democritus περὶ λογικῶν καὶ κανών, which has
been preserved by Sextus,[2] two kinds of knowledge are
distinguished: one relative to the understanding (διὰ τῆς
διανοίας), the other to sensation (διὰ των αἰσθήσεων):

"There are two forms of knowledge, one pure or gen-
uine, one obscure or spurious. To the obscure belong all
of the following: sight, hearing, smell, taste, touch. But
the pure form is entirely different." And he adds that this
pure knowledge is relative to a finer organ of thought,
which takes the place of sight, or hearing, or smell, or
touch in the smallest things (connecting us in this way
with the true nature of things, namely, with the atoms).

The relation between the two forms of knowledge is
expressed clearly by Democritus also in the following
statement: [3] "By convention (νόμοι) sweet is sweet,
by convention bitter is bitter, by convention color

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[1] *Metaphysics* XI, 4 (3), *De Partibus Animalium* I, 1 (ed. Didot, vol.
III, pp. 223, 2).

[2] In Diels, B. 11.

[3] Galen, (in Diels, B. 125); cf. Sextus (in Diels, B. 9).

is color; only the atoms and the void are' real."
But then he makes the senses speak against thought, and
adds: "Poor mind, it is from us thou hast got the proofs
to throw us with. Thy throw is a fall." [1]

This remark is extremely interesting. We see thus that
Democritus discussed, like Plato and Aristotle, and pre-
viously to them, the problem of the origin of ideas. But
instead of stopping, like the Athenian philosopher, at the
assumption of inborn knowledge (theory of reminiscence)
he rather seems to derive ideas from sensations. It is in
this way legitimate to suppose that Aristotle may have
borrowed from him the view which we saw him express
in the Posterior Analytics II, 15.

It is true that in the case of Aristotle it is difficult to
see how this doctrine could be reconciled with the dignity
attributed to the inductively acquired notions which have
to serve as necessary premises for demonstrative science.
As regards Democritus, however, we are able to solve
the difficulty by what we know of his theory of sensations
(in connection with the fundamental atomic supposi-
tion). Our philosopher indeed believed [2] that sensations
in general are caused by small images (εἴδωλα) emitted
by bodies and capable of impressing themselves upon
our sense organs and even upon thought itself in the same
way as light affects a photographic plate. The images
which correspond to thought knowledge come directly
from the atoms and are of a finer nature. It is therefore
conceivable that they can disengage themselves from
the union with the coarser images striking the senses in
those cases where a comparison of repeated sensations

[1] The translation is that of Burnet.—Note of translator.
[2] Cf., for instance, Aetius (in Diels, A. 30).

received from numerous objects enables us to establish the common characters which define the concept.

3) That Democritus recognized the logical value of concepts almost as an anticipation of experience can also be seen from the testimony of Diotimus in Sextus (VII, 140I).[1] We are told that he assumed "appearance as a criterion for the understanding of obscure things, and the concept as a criterion for research": ἔννοια κριτήριον ζητήσεως.

The use of the term ἔννοια is interesting here. We have already noticed it in connection with the designation κοιναὶ ἔννοιαι applied by Euclid to axioms, and we have pointed out that this term is not to be found in the philosophical literature of Plato and Aristotle and that it appears later among the Stoics. Plutarch seems to refer in Olympiodorus[2] to a work of Chrysippus περὶ ζητήσεως when he says that "the Stoics alleged natural concepts τὰς φυσικὰς ἐννοίας as a basis for this (that is, for the possibility of arriving at things which we do not know). On the other hand, we are told by Diogenes Laertius (VII, 54)[3] that according to Chrysippus "there are two criteria of truth, sensations and concepts." The place of the word ἔννοια is taken here by the term πρόληψις, which also occurs in the works of the Epicureans and which denotes "anticipation (of experience)."

We can obtain the precise meaning which the Stoics attached to the term ἔννοιαι from a passage in St. Augustine's *De Civitate Dei*,[4] where the Stoics and Epi-

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[1] Diels, A. III.

[2] Cf. Arnim, *Stoicorum veterum fragmenta*. Vol. II, no. 104. Chrysippus was a pupil of Zeno of Cition (280-209 B.C.).

[3] In Arnim, op. cit. 105.

[4] In Arnim, 106.

cureans are spoken of as those philosophers who base truth on the senses:

"Qui cum vehementer amaverint sollertiam disputandi, quam dialecticam nominant, a corporis sensibus eam ducendam putarunt, hinc asservantes animum concepire notiones, quas appellant ἔννοιας, earum rerum scilicet quas definiendo explicant . . ."

These references seem to entitle us to infer that the Stoics adopted, like Aristotle, the Democritean doctrine of the sense origin of concepts (the Epicureans alone clung to the assumption of small images). They, however, robbed concepts of the sublime dignity which rationalists try to confer upon intelligible entities. Scientific demonstration ἀπόδειξις thus reduces itself for them, as Cicero says,[1] to a "ratio, quae ex rebus perceptis ad id, quod non percipiebatur, adducit."

In connection with these views, which are of a more empirical nature, it is interesting to point out how the Democritean doctrine of knowledge becomes modified. Zeno of Cition, for instance, says that it is "a sure, firm and immutable comprehension by reason" (ἀμετάθετον ὑπὸ λόγου κατάληψιν) or also "an immutable possession of reason in the apprehension of ideas" (ἐν φαντάσιων προσδέξει).[2]

The Stoics thus did not arrive at the pure empiricism of Epicurus, who regarded every sensation and appearance as true. Appearance had to be conjoined, according to them, with the voluntary assent of the soul,[3] which for the wise man is based on the identity existing be-

[1] Arnim, III.

[2] The references are from Sextus and Diogenes Laertius (in Arnim, Zeno of Cition, no. 68).

[3] Cf. Sextus and Cicero (in Arnim, Zeno of Cition, no. 63 and 61).

tween individual and universal reason or logos. By appropriating the Heraclitean "logos," the Stoics were able to preserve for thought a certain dignity and in this way to facilitate the transition to the later views of the eclectics (Cicero). For the latter *common notions* were no longer considered as principles expressive of the uniformity of nature but as *inborn ideas* bearing testimony to the reminiscence of man's divine origin. The Stoic theory (which in reality goes back to Plato) merges here into Neoplatonism.

The Epicureans are connected with Democritus in a more direct way than the Stoics (who derived from him, to be sure, the principle of universal determinism). It is they who adopted his atomic theory, which they, however, robbed of its deeper mechanistic significance. But as we have pointed out already, Epicurus (341-270 B.C.) is far from the rationalism of the master of Abdera. His Canonic contains a few rules which are clearly expounded by Sextus Empiricus and which were restated with precision by Gassendi in his logic.[1]

Here are the most essential of the Epicurean rules:

    I. Sensus nunquam fallitur.

    II. Opinio est consequens sensum, sensionique superadiecta, in quam veritas aut falsitas cadit.

    III. Opinio illa vera est, cui vel suffragatur, vel non refragatur sensus evidentia.

    IV. Omnis quae in mente est anticipatio, seu praenotio, dependet a sensibus; idque vel incursione,

[1] Petri Gassendi, *Opera Omnia* (Florence, 1677), vol. I, part I. De Logica Origine et Varietate.

vel proportione, vel similitudine, vel compositione.
(This manner of forming concepts appears in
the Stoics.)

V. Anticipatio est ipsae rei notio, sive definitio.

VI. Est anticipatio in omni ratiocinatione princi-
pium . . .

VII. Quod inevidens est, ex rei evidenti anticipatione
demonstrari debet.

What is interesting here is the appeal to sense evidence
(ἐνάργεια), which is regarded as a criterion of truth. It
is easy to recognize in this the Democritean criterion,
despite the changes undergone by it. In opposing pure
and genuine knowledge to obscure knowledge the
Abderite exactly made clearness of ideas the sign of their
value. Only what was for Democritus conceptual clear-
ness becomes for Epicurus sense clearness.[1] Nineteen
hundred years later Descartes will come back to the cri-
terion of intellectual evidence and he will find in *clear-
ness* and *distinctness* a mark of true ideas (the addition
"distinctness" comes from the *Theaetetus* 209c-210).

7. THE LOGIC OF THE SKEPTICS

After having spoken about the Stoics and Epicureans
we have to turn now to the Skeptics. It is true that these
thinkers did not form a sect or a well defined school.
They present, however, beginning with Pyrrho of Elis
(about 365-275 B.C.) and his friend Timon, a certain
continuity of critical tradition which expresses itself in

[1] It is noteworthy that the criterion of evidence is applied already by
Theophrastus to thought as well as to sensation (cf. Sextus, *Adv. Math.*
VII, 217).

an attitude of methodical doubt maintained towards dogmatic philosophical systems. Arcesilaus of Pitone (about 315-241) and Carneades (who became in 155 B. C. ambassador in Rome) introduced Skepticism into the Middle Academy. Later we meet with Aenesidemus of Knossus (who probably lived in Alexandria at the beginning of the Christian era), a century later with Agrippa, and finally with Sextus Empiricus (third century A.D.), who summed up the whole movement in a valuable work which forms a most important source of information for the history of Greek thought.

That Skepticism was to a certain extent influenced by the philosophy of the Abderite can be seen from the external relation which, according to tradition, existed between Pyrrho and certain disciples of Democritus, like Nausiphanes, as well as from the skeptical tendencies ascribed to other followers of Democritus (Metrodorus, Anassacrus). But the link between the two systems is above all to be found in the moral motive which inspired the reserve assumed by the Skeptics towards the true nature of things. For the aim of the suspension of judgment was the attainment of that "ataraxia" or peace of mind which in the end reduces itself to the mastery of the passions advocated by Democritus.

But the theoretical connection between Skepticism and the teaching of Democritus follows from the fact that the latter had reduced reality to the indifferent matter of the atoms and denied the external existence of sense qualities. By a further step of criticism (going back to the point of view of Protagoras) the doubt had naturally to be extended also to those primary qualities in which the great atomist had discovered the intelligible object

of knowledge. This development was certainly suggested by the contrast between the views of the two rationalists who had made it their aim to combat the empiricism of Protagoras: Democritus and Plato. The latter precisely regarded as intelligible the very qualities (hypostatized under the name of ideas) which were for the former empty appearances. Besides, it is possible to detect in the Democritean system the origin of the attack on intelligible entities if it is true that our philosopher derived the understanding also from the senses, as we are led to believe on the basis of an inductive study. The road traversed by Greek thought does not seem in this way to be unlike the one by which modern philosophy finally arrived, through the ideas of Galilei, Descartes, and Locke (in all of whom the distinction between primary and secondary qualities reappears), at the criticism of Berkeley. Starting from the theory of vision the latter goes as far as to deny also the transcendent significance of this geometric substratum of matter.

We have to point out that Skepticism does not reject entirely the phenomenal world. Its attack is mainly directed against the dogmatists who assert that we can reach something of the truth or of the nature of things in themselves. By exposing the relative elements contained in the criteria of truth, this criticism represents a durable acquisition for the theory of knowledge. It will be found that the spirit animating it has great affinities with that of modern positivism if we abstract from the sentiment with which a more advanced science inspires the critics of metaphysics to-day.

But for the history of logic it is especially important to examine the arguments of Carneades, which are to

be found in Sextus Empiricus,[1] against the Aristotelian conception of demonstration. We meet here again with the view, put forward by the predecessors of Aristotle and combated by the latter, that every demonstration leads to an *infinite regress*, because every premise has to be deduced from another premise. This argument derives its force from the denial of all immediate certainty. As we have seen, the Skeptics were led to this conclusion by the view that the concepts of our judgments have their origin in sense perception so that the uncertainty of sensation is communicated also to the understanding. This is why they examine the doctrine which asserts that it is legitimate to base science upon hypotheses which are to be rendered in their turn firm and valid by the consequences that can be deduced from them. The passage in Sextus where this doctrine is criticized [2] does not say who was responsible for it. But it is quite clear that it especially reflects the views of the mathematical physicists. And there is also perhaps some reason for ascribing it to Democritus, who was the first to assign to science the task of explaining phenomena rationally. We indeed have already pointed out that it may well have been the Abderite whom Aristotle had in mind when he argued that the attempt to prove premises by means of conclusions leads to a vicious circle.[3]

Carneades takes up again the Aristotelian thesis when he asserts that the false can be deduced from the true. From a strictly logical point of view this argument can certainly not be refuted. But although the Skeptic is in-

[1] *Adv. Math.* VII, 159-189, and VIII, especially 367-463.
[2] *Adv. Math.* VIII, 375.
[3] Loc. cit., An. Post., I, 2 (6).

clined to lay special stress on this negative assertion, Carneades does not stop there. After denying the existence of absolutely certain criteria for both the true and the false, he grants, nevertheless, probable value to knowledge. He ascribes this value primarily to every idea possessing sufficient evidence, but in a higher degree to the concatenation of ideas in a logical system (ibid., VII, 375 ff). In the last analysis this does not differ greatly from the positive criterion by which we determine to-day the value of scientific theories. But we nowadays display in this respect a more trustful attitude, which is due to the development of the mathematical treatment of physics. The sentiment of the Skeptics, on the other hand, corresponds to a less evolved science, and also to a mode of thinking characteristic not so much of mathematicians as of the medical circles, in which ancient Skepticism was popular. In reality the employment of hypotheses the probable value of which is derived from the experimental verification of their consequences is the characteristic trait of the deductive and experimental method of modern science, as we shall find later in Kepler, Galilei, and Descartes.

We have tried to find traces of pre-Aristotelian ideas in the post-Aristotelian logic. Our examination has revealed that Greek thought finally transcended the *logical realism* of Aristotle and that it reached positions corresponding to the most advanced modern views. Our knowledge of the critical work of the post-Euclidean geometricians is too meager to enable us to gauge its importance. From what we know we may infer that Apollonius found

no followers in his acute geometrical researches. On the other hand, the works of the Hellenistic philosophers who reflected on the nature of science were not connected with the scientific development proper and much less with the mathematical movement, and therefore assumed often that negative form which found its most refined expression in Skepticism. For observers who were unable to resume and continue the profound thought of the early mathematical philosophers the refutation of Aristotle's belief in an order of necessary truths was tantamount to the denial of the possibility of all science.

We find, however, an interesting case in the Stoic school, where a formal treatment of logic was combined with an empiricistic doctrine of knowledge. And if this formal development ended in an arid schematism (which seems to account for the contempt of the Stoic Aristo of Kos quoted at the beginning of the present work), we must not, for this reason, overlook the importance of logico-grammatical researches which enable us to detect in language a certain expression of the constructive activity of thought. We do not wish to examine here to what length this analysis was carried on by the Stoics. But it is certainly in their works that one can discover that distinction between the subjective and objective which will be elaborated in the religious turmoil of the Christian soul to reappear at the beginning of the modern era as the basis of philosophy.

8.   BRIEF REMARKS ON MEDIEVAL LOGIC

From Greek history we shall pass to the dawn of the modern period, ignoring the movement of ideas by which

the rebirth of science was accompanied. It will be suffi-
cient to point out that the general character of the de-
velopment of logic in the intermediary period was arid, if
not entirely barren. We may say in this connection that
the Aristotelian and Stoic logic was introduced among
the Romans by Boethius (470-525 A.D.). His transla-
tions of Aristotle's Categories and On Interpretation,
of Porphyry's *Isagoge*, together with the commentaries
with which he himself as well as other Neoplatonic
authors provided these writings (in the sense of formal
technique, according to the Stoic tradition), formed the
cultural basis of the early Middle Ages in this domain.
Apart from this, the general culture of this period was
represented by a certain number of encyclopedias coming
from late antiquity, for instance, the one by Martianus
Capella (in the fifth century). These works deal with the
so-called seven liberal arts, which formed in the scholastic
curriculum the trivium (grammar, rhetoric, and logic)
and the quadrivium (geometry, arithmetic, astronomy, and
music). It is especially noteworthy that neither the other
works of Aristotle (logical, physical, etc.) nor the original
works of Plato, with the exception of the *Timaeus*, which
was translated from the Latin by Calcidius, were known
in the early Middle Ages. A more extensive knowledge
of these works as well as of the classical scientific move-
ment in general was brought about in Europe through
contact with the Arabian civilization, the influence of
which began to be felt in the twelfth century (Petrus
Hispanus). Later the humanistic Renaissance was to reap
rich fruits from a direct acquaintance with Greek texts,
made possible by the numerous Greek scholars who came

to settle in Italy as a result of the fall of the Eastern empire.

Two aspects are of importance in the scholastic logic:

1) The progressive elaboration of the formal technique, which became especially refined owing to the subtle distinctions of Arabic and Byzantine origin.

2) The great question of the reality of universals; its dramatic character is hardly revealed by the barren schematic form of the discussions.

We shall pass over the first point, although it would be of interest for the history of logic to show, for instance, that Buridan (died about 1360) was aware of the distributive property of the particle "non" with reference to the particles "and" and "or":

$$\text{non (a and b)} = \text{non a or non b}^{[1]}$$

A similar analysis might be found also in Paulus the Venetian.

As to the problem of universals we should like to say that we have here to do with an old question that originated in the Platonic-Aristotelian ideology, namely whether there is a reality outside the human mind corresponding to general ideas. This question was brought to life again by a passage in Porphyry's *Isagoge* (I, 3):

"And above all, as regards genera and species I shall not examine the question whether they exist by themselves, or whether they exist as pure concepts of the mind only; and—in case they exist by themselves—whether they belong to corporeal or incorporeal things; and finally whether they have a separate existence or in sense things

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[1] I owe this to my friend Vacca.

only. This is too deep a question and would require an extended study."

It seems that in the vast network of the medieval polemic the *nominalists* (denying the reality of universals) in general represented scientific tendencies against the Platonizing mysticism of the *realists*. This is especially true of the fourteenth-century nominalists, William of Occam (died 1347) and John Buridan, rector of the University of Paris, who originated the theory known as *terminism*. This theory (which reminds one of Abelard's *conceptualism*) regards concepts (*termini*) as subjective *signs* (*signa*) of singular things, or of classes of really existing things. Logic thus deals only with these signs—whether written, spoken, or conceived—of things (Occam, *Quodlibeta* V, 5). Occam avers that the concept acquires its proper meaning in the proposition and often through a combination with some other term: "terminus conceptus est intentio seu passio animae aliquid naturaliter significans, aut consignificans, nata esse pars propositionis mentalis."

While this doctrine transcends a narrow nominalism, it, however, rejects realism. It denies that the real meaning of a concept is to be found in its *comprehension* or *connotation*, that is, in all the *marks* or attributes whose substantial unity it is supposed to express. It attaches itself instead to *extension* or *denotation*, that is, to the sum total of the objects for which the concept stands and which are truly unified in the human mind in virtue of certain real similarities.

Scholastic definitions, which descend from the general to the particular, and logic itself lose their importance in the light of this view. As a result verbal explanations

were to give place to concrete experience. This explains sufficiently the passionateness with which the polemic about universals was carried on. Translated into social and moral terms, it amounted to a vindication of individual freedom, to a protest against the tyranny of institutions, the authority of articles of faith and of traditional instruction. To attack the tree of sterile deductions at its root, to reconstruct all knowledge upon an inductive basis, appeared as the best means for bringing about such an intellectual emancipation. The same tendency manifests itself to a larger extent in the anti-Aristotelian reaction of the humanists, who set out to purify logic from the scholastic subtleties (Valla, Agricola, Vivès), and finally it assumes new forms in the rebirth of the scientific movement.

II

RATIONALISM AND THE EVOLUTION OF MODERN LOGIC

9. BACON AND INDUCTIVE LOGIC

The development of modern thought was motivated by scientific aspirations which could no longer be satisfied with old scholastic schemes. Reflective minds began to revolt against verbal explanations which resorted to occult qualities, so brilliantly satirized by Molière:

> quare opium facit dormire?
> quia habet virtutem dormitivam.

The appeal was being made now—as we have already pointed out—to turn from the barren manner of syllogizing to concrete experience.

Hence the tendency, which comes to light with the dawn of modern science, to proceed inductively from the particular to the general. But this tendency not only manifests itself in the attitude of those who approach experimental data from the point of view of induction proper; it also colors the meaning which the great thinkers attach to deduction. Tradition certainly narrows the meaning of deduction in regarding it as a pure descent from the general to the particular. And it is therefore hardly adequate to characterize the progress of modern thought as based on an opposition between the classical deductive logic of Aristotle and the new inductive logic which Bacon is supposed to have founded in the *Novum Organum Scientiarum* (1620) and which is supposed to

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have been perfected and systematized by John Stuart Mill.

In order to arrive at generally valid knowledge, we have, according to Bacon, to compare particular observations or experiments and to take into account not only similarities (as is the case with Aristotle's "simple enumeration") but also negative instances. This method, however, does not yet quite correspond to the experimental thinking, as employed in modern physics.

The English philosopher, who followed in the footsteps of Bernardino Telesio, realized quite well how the interpretation of nature "a sensu et particularibus excitat axiomata, ascendendo continenter et gradatim, ut ultimo loco perveniatur ad maxime generalia" (op. cit. I, 19). But he was wrong in conceiving this ascending development of thought to be in opposition to the method of the most general "anticipationes naturae" from which the "axiomata media" are deduced. For this deduction has exactly to be regarded as a constituent part of the inductive process itself. The author of the *Novum Organum* manifests here an insufficient insight into the value of quantitative determination. And although he had some inkling of the function of hypotheses—he was aware of their clarifying value (citius emergit veritas ex errore quam ex confusione)—he does not seem to have grasped the full importance which they can assume through the use of mathematics. Francis Bacon of Verulam (1561-1626) proclaimed himself the herald ("buccinator") of the scientific rebirth. In reality he remained behind the ideas which earlier thinkers, like Leonardo da Vinci [1]

[1] Cf., for instance, the maxims 1148-1160 in J. P. Richter, *The Literary Works of Leonardo da Vinci* (London, 1883), vol. II, p. 288.

(1452-1519), had anticipated in a more comprehensive manner and which contemporary scientists, like Kepler (1571-1630) and Galilei (1564-1642), splendidly realized.

It is interesting to point out that not only did not Bacon himself make any observations or experiments but that he also often remained enthralled in the scholastic world of ideas. In his pseudo-induction, for instance, we see him advocate a method which is reminiscent of that of the alchemists. According to this, we have only to discover the "forms" or simple qualities of things (naturae simplices)—thick, thin, white, etc.—and the things themselves will follow from these properties by composition. Bacon also attempts to give an interpretation of Plato's doctrine (Cf. *De augmentis scientiarum*). We find here a conception which goes perhaps back to the Anaxagorean view of matter. Tannery indeed saw in this the origin of the theory of ideas.

A friend of mine, a Sanskrit scholar, told me one day the following saying of a Hindu philosopher: There are three kinds of great men: some are born great, some become great, some are made great. We hope not to be guilty of irreverence by putting Bacon in the third category; for his greatest successors in the field of British empiricism always extolled him as the first of their countrymen who gave expression to their tendencies at the beginning of the new era.

With these remarks we do not intend to underestimate the services which Bacon's *trumpet-call* (like the martyrdom of his predecessor Petrus Ramus) may have ren-

dered to the destructive battle that was waged for the salvation of science. Apart from the vividness of his style, he certainly deserves credit for having made himself the mouthpiece of the dominant scientific ideas of his day, and especially for having advocated views which the philosophical reaction of the nineteenth century blamed as unhistorical. We hear him, for instance, proclaim the greatness of the Greek philosophers who preceded Plato and Aristotle, especially of Democritus,[1] indulging in phrases which are far from reverential: "Not Plato, not Aristotle destroyed that philosophy, but Genseric and Attila." "After the shipwreck of this doctrine the Platonic and Aristotelian philosophical planks were saved and transmitted to us owing to their greater lightness and inflatedness." Our philosopher gives here expression to the intimate conviction which the scientists of his epoch derived from an exact study of Aristotle. Through the latter as well as through other sources of the classical tradition they were led to regard the early Greek thinkers as the immediate forerunners of their own work.[2]

It is to these scientists—who are in reality and in a higher sense also philosophers—that we shall turn for enlightenment in regard to the new conception of the logic of science, in which deductive development always plays the largest part. We shall have to proceed on this

[1] "De principiis atque originibus secundum fabulas cupidinis et coeli, sive Parmenidis et Telesii et praecipue Democriti philosophia tractata in fabula."

[2] I cite as an example a letter from Kepler to Galilei, dated April the 19th, 1610, where he speaks about the hypothesis of the infinitude of worlds "which was favored by Democritus, Leucippus, and among modern thinkers by Bruno and Bruzio, our common friend" (*Opere di Galileo,* III, p. 106, and X, p. 321).

road to the end, up to the reform of the Aristotelian logic
in the nineteenth century, to be able to estimate properly
the value of the so-called inductive logic (which really
deals only with the lower forms of scientific construction)
in comparison with the true inductive aspect of a more
advanced science. We shall find then that the concept of
progress has taken the place of the ideal of a demonstra-
tive science based on immutable principles, which thus
make science itself immutable. Scientific constructions
will then be seen to be erected not so much upon first
principles which are immediately suggested by simple ob-
servations, as upon principles which always follow as
consequences from the preceding scientific development.
For what we deduce from provisionally accepted hypoth-
eses leads through experimental verification to their criti-
cism and renovation.

10. THE RATIONALISTIC CONCEPTION OF SCIENCE IN KEPLER AND GALILEI

Two different motives seem to have exercized the great-
est influence on the development of the concept of logic
at the beginning of the modern era:

1) The extension of mathematics, which came to em-
brace beside the astronomy and the geometry of the
ancients also mechanics and physics;

2) The development of the operational symbolism
(first algebraic, and then infinitesimal).

As to the first point it is obvious that the ideal of a
science of becoming had to declare war against the Pla-
tonic-Aristotelian realism, inspired as the latter was by
a static conception of science shaped after the pattern

of geometry. But let us examine a little closer the views which the founders of modern astronomy and mechanics entertained, in part also explicitly, in regard to logic.

It is well known that Kepler freed himself only gradually from the prejudices and confusions in which he was enthralled during the first part of his life. Only after long years of intellectual endeavor he was able finally to arrive at the precise scientific ideas which determined his immortal discovery of the laws of planetary motion. This development has more than a psychological interest. It can be partly explained by the fact that he seems to have derived the first stimulus to speculate from the Pythagoreans. In these thinkers he indeed found that mixture of comprehensive views and mystic ideas which was so difficult for his scientific thought to overcome. In his *Mysterium Cosmographicum,* for instance, he still looks for the explanation of the structure of the universe in those regular geometric bodies which the Pythagoreans surrounded almost with a religious veneration.

We see, nevertheless, Kepler arriving in this work at interesting logical considerations. In the first chapter, where he discusses the Copernican system, he repudiates the views of those who follow Aristotle in denying that premises can be proven by conclusions "freti exemplo accidentiae demonstrationis, quae ex falsis praemisis Syllogistica verum aliquid infert." He resorts to probability to show that the deduction of something true from something false is a matter of chance. By starting from false premises we shall get involved sooner or later in erroneous consequences, unless we successively assume an infinite number of other false propositions.

In the first book of the *Epitome Astronomiae Coperni-canae* [1] Kepler describes the logical procedure of science in a much clearer way. Starting from preliminary observations, it rises inductively to hypotheses that are apt to save appearances. The value of these is then to be established by examining their consequences in the light of geometry, physics, and metaphysics. What is interesting here is the rationalistic character of the Keplerian point of view. He makes the choice of hypotheses depend not on their agreement or disagreement with observations (the phenomena of celestial motion can indeed be equally well explained either by the hypothesis of Ptolemy, or by that of Copernicus, or by that of Tycho Brahe) but on their conformity with other *reasons:* "Non enim debet esse licentia Astronomis fingendi quidlibet sine ratione." We thus see how Kepler is still guided at the height of his scientific development by Pythagorean criteria, better understood, of course, despite the fact that he had been led by his own observations and by those of Tycho to lose faith in the possibility of determining the laws of the universe exclusively by a priori speculations (by basing himself on Pythagorean polyhedra).

The same rationalistic conception of science is to be found in the background of the thought of the greatest Italian philosopher and scientist, Galileo Galilei. It is true, his connection with rationalism is often overlooked owing to the fact that he was the founder of the experimental method. The very origin of that method can, how-

[1] And already many years earlier in an unfinished work, *Apologia Tychonis contra Ursum.*

ever, not be properly understood apart from this rationalism.

Galilei was led from his early youth, through a study of Aristotle, to reflect on the nature of demonstrative logic. This can be seen from a manuscript which the editor of the national edition of the *Opere* did not publish in full, since he regarded it as a school exercise simply copied by Galilei; the editor has, however, given an instructive summary of it.[1] Several interesting questions are to be found there: for instance, whether the knowledge of premises is greater or more perfect than that of conclusions, whether the principles of science are known independently of any proofs, or whether they admit of a demonstrative regress, etc. But even if we accept the opinion of the editor that the manuscript was an exercise dictated to Galilei by some one else, the fact still remains that these questions must have occupied the pupil's mind. We may then infer that they continued to interest him in his mature days.

We cannot enter here upon a detailed examination of the works of Galilei. This would show that our author started his philosophical career with an attempt to explain and criticize Aristotle, and that he soon became interested in the views of the philosophers that were criticized by the Stagirite and notably in those of Democritus. Although he was led by motives of prudence not to speak openly of the latter, he was none the less reproached by adversaries for following his doctrine. And as a matter of fact he developed, as we shall see, not only the principles of the Democritean physics but also some of his fundamental philosophical views.

[1] *Opere*, IX, 280-81.

The rationalistic criterion of science shows itself in all the works of Galilei, which always tend to deduce phenomena mathematically from plausible hypotheses. He says himself, for instance, in the notes to the *Esercitazioni philosophiche di Antonio Rocco* [1] that he became "convinced by reason *before* he was assured by the senses" that the velocity of falling heavy bodies is independent of their mass. For he had "laid it down as an indubitable axiom" that the combination of two equal falling masses cannot increase their velocity.

This principle of sufficient reason appears also implicitly in his other demonstrations. It is true that he often appeals to observation and experiments in order to settle controversies with his adversaries. A closer examination, however, discloses that the manner of this appeal is not in disagreement with his rationalism. On the contrary, sense evidence is invoked by the Peripatetics against him rather than by himself. Thus in the *Dialogo sopra i due massimi sistemi del mondo* it is Simplicius who refers to the view of Aristotle when he says "that sense perception has to be preferred to every discourse spun out by the human mind and that those who deny the importance of a certain sense deserve to be deprived of that sense." [2] Salviati replies to this that Aristotle would have changed his views, if he had been familiar with the new observations. He adds the following characteristic remarks: "Do not doubt that Pythagoras was certain that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the two legs long before he discovered the demonstration for which he sacrificed the hecatomb. When the conclusion is true, it is

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[1] *Opere*, VII, 731. [2] *Opere*, VII, 75.

easy to arrive by means of the regressive method at some proposition that has been already demonstrated and to strike upon principles that are more known." "Demonstrative sciences," says Salviati, "generally proceed by using first the senses to make sure, *as far as it is possible*, of the conclusions," etc., etc.

These considerations would be ascribed to the influence of Archimedes, had his work on Method been known at that time. They show that Galilei's ideal was to find a true demonstration based on rational principles independently of the preliminary observations whose function it is to indicate the direction of research. The distinction between experiment and demonstration can be seen from the following words which the author puts into the mouth of Salviati in the same dialogue: "A single experiment or conclusive demonstration to the contrary is sufficient to defeat a thousand probable arguments."

Galilei himself, moreover, already had said "that in every demonstrative science it is requisite to lay down the definitions of the terms proper to each discipline as well as the chief assumptions, which are to prove to be fruitful seeds giving birth to the causes and true demonstrations of all the properties of mechanical instruments." [1] In another place he had maintained in connection with sense error that deception cannot be admitted as a general principle, and that the greatest sense errors can be corrected by mathematics, since "deception lies not in the 'sensus communis' but in the separate senses." [2] On the whole he seems to adopt the views of Democritus and Aristotle in regard to the origin of general ideas.

Galilei's rationalism reveals itself especially in the dis-

[1] *Opere*, II, 159.   [2] *Opere*, III, 321.

cussion of the properties of matter, in which "Il Saggiatore" follows Democritus. He says: [1] "As soon as I conceive matter or a corporeal substance I feel compelled to conceive at the same time that in relation to other bodies it is large or small, that it is in this or that place, at this or that time, that it is in motion or at rest, that it touches or does not touch another body, that it is one, several or many; by no act of the imagination can I separate it from these necessary conditions. But I feel no compulsion to think that it must necessarily be white or red, bitter or sweet, sonorous or mute, of a pleasant or unpleasant taste. If the senses did not guide us, imagination and discourse would perhaps never arrive at these sensations by themselves. It therefore seems to me that these tastes, odors, colors, etc., are nothing but names in regard to the things in which they seem to inhere; they reside solely in the perceiving body. . . . But I do not believe that anything else is requisite in external bodies beside magnitude, figure, multiplicity, and slow or rapid motion in order to call forth in us tastes, odors, and sounds."

It is precisely from the intuition of the simplest properties of these magnitudes and figures and their motions that Galilei wants to derive the principles of demonstrative science so as to be able to deduce from them all more hidden truths. Such a procedure seems to him to be a pale image of the intuitive knowledge possessed by God. "The divine intellect grasps with the simple apprehension of his essence and without having recourse to discourse in time all the infinity of these properties (of a circle), which are virtually contained in the definitions of

[1] *Opere,* VI, 347.

all things, and, infinite as they are, they are perhaps one in their essence as well as in the divine mind. This is not entirely unknown even to the human mind. . . . What our intellect accomplishes in time and step by step, the divine intellect reaches instantaneously after the fashion of light; in other words, it has everything present at once." [1]

What value did Galilei then ascribe to experiments to which, as we have already seen, he resorted in his controversies with adversaries who were unable to follow his reasons?

An experiment was for him a test which the true doctrine meets, a means for establishing the fact that nature cannot fail to agree with the expectations of reason. It was for him in short a way of showing that truth possesses sufficient light to pierce the darkness of error and enough strength to resist all the efforts of false opinion to appear as true. This is the significance of the most famous experiments of Galilei and his disciples, for instance, those of the barometer made by Torricelli.

An experiment had to appear as all the more significant, since the experimenter was convinced that "an effect can have only one true and primary cause," [2] all the others being false and fabulous.

The purpose of an experimental proof was thus for Galilei negative rather than positive. Demonstrative logic was in this way brought back to the origins of dialectic, since the experimental method assumed the character of a *reductio ad absurdum*. This analogy has already been pointed out by Duhem and Vailati.

The view that experiments can by themselves form

[1] *Opere,* VII, 129.   [2] *Opere,* VII, 447.

premises for research forced itself only gradually upon
Galilei's school. A letter by Baliani to Galilei dated
July 1, 1639,[1] is interesting in this connection. "I am
of the opinion that experiments possessing certainty have
to be laid down as the principles of science and that it
is the part of science to lead us from the things derived
from the senses to the knowledge of those things that
are unknown." And after referring to the apparent
reserve of Galilei, who promises to deal with these mat-
ters some other time, Baliani goes on to say: "Since the
principles of sciences are definitions, axioms, and pos-
tulates, I think that in the natural disciplines these are
for the most part experiments upon which astronomy,
music, and mechanics are based." The distinction made
by the ancient Greeks between axioms and postulates has
come in this way to be applied no longer to the premises
of geometry but to those of mathematical physics. This
naturally gave rise to the old discussions, namely
whether postulates could be eliminated altogether by
basing science exclusively upon a priori principles, or
whether the distinction itself ought to be given up in
the light of a more detailed examination which would
reveal the experimental nature of all principles.

11. THE LOGIC OF DESCARTES

The views maintained by Kepler and Galilei in regard
to the part played by hypotheses in determining the value
of deduction are expressed clearly by Descartes at the
end on the *Discours de la Méthode*.[2]

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[1] *Opere di Galileo*, XVIII, 69.
[2] *Œuvres*, ed. Adam et Tannery, VI, 76.

"If some of the matters about which I spoke at the beginning of the *Dioptrique* and the *Météores* seem strange at first sight, because I call them hypotheses and because I do not try to prove them, I beg the reader to go through everything carefully; I hope that he will then be satisfied. For it seems to me that the reasons are interconnected here in such a way that the last are proven by the first, which are their causes, and the first are proven in their turn by the last, which are their effects. Nor can I be accused of committing the fallacy which logicians call a circle; since most of the effects are rendered very certain by experience, the causes from which I deduce them serve not so much for proving as for explaining them. On the contrary, it is the causes which are proven by the effects."

When Descartes (1596-1650) wrote these words, he had just passed the age of forty and had succeeded in formulating the principles of his rationalistic philosophy, to which he gave a new expression in 1641 in the *Meditationes Metaphysicae*. The importance which he bestows upon experiments reveals the part which the ideas of Bacon played in the development of his thought. His correspondence with Mersenne bears testimony to this. He says indeed in a letter of December 23, 1630, in reply to a question about a possible way of making useful experiments: "I have nothing to add to what Bacon of Verulam has already said in regard to these things. I may only say that I find it important to have general collections of all the most common facts, although I am not interested in finding out all the small

details concerning any particular matter . . . For instance, the fact that all shells are coiled in the same direction, and whether this is true on the other side of the equator." In a letter of April 5, 1632, he writes that he hopes to establish in time all the things for which he had tried to pave new ways "by adding experiments to ratiocination." And on the tenth of May of the same year he refers again to the method of Bacon of Verulam which describes the heavens without resorting to hypotheses or causes.

A closer examination would, however, reveal that the value of experiments consisted for Descartes especially in the fact that they stimulate reflection and that they lead by means of a regressive and analytic method to the discovery of the primary causes of things from which the synthetic explanation of science has to start. The skeptical arguments which occupied his mind considerably [1] convinced him of the fallaciousness of the senses. The only reliable criterion that appealed to him here was the natural light of reason, that is, clearness of thought.

It is well known how Descartes made methodical doubt the starting point for his speculations; how he believed that he had overcome this doubt by finding in consciousness the proof for his own existence; how he arrived through this Augustinian principle at the a priori demonstration of the existence of God and restated the ontological argument of St. Anselm in a slightly modified form; how he derived from the veracity of God confidence in the natural light of reason in which he had doubted for a time. We shall not enter upon these theo-

[1] Cf. *Œuvres,* I, 353.

logical arguments. They are useless for an understanding of the Cartesian logical criteria.

The clearest exposition of Descartes' logic is to be found in the *Regulae ad Directionem Ingenii*. (*Œuvres*, X, 339-469.) We shall especially quote the following rules:

Rule III. In regard to the objects one intends to study one shall look not for the opinion of others or for one's own conjectures but for what can be established clearly and evidently in intuition, or for what can be deduced with certainty; for science cannot be acquired otherwise.

Rule V. The whole method consists in the order and arrangement of the things to which we have to turn our mind, if we wish to discover a certain truth. We shall follow it exactly by gradually reducing obscure and complicated propositions to simpler ones, and by trying to rise by the same steps through the intuition of the easiest things to the knowledge of all others.

The sixth rule explains how to distinguish between simple and complex propositions. In the eleventh rule we are told to pay the greatest attention to the easiest and least important things, until we succeed in seeing truth clearly and distinctly. The seventeenth rule asks us to establish intuitively the interdependence of terms "per veros discursus" and to abstract from the question whether they are known or unknown. Apart from these norms whose aim is the logical systematization of knowledge, we are also advised to make use of memory, imagination, the senses (XII), and to draw the figures to which our judgments refer (XV), but only for the purpose of stimulating and retaining the attention of the mind.

It is interesting to refer in connection with these rules to the Cartesian attitude towards the "consensus gentium," which is quite clearly expressed in a letter to Huygens of October 16, 1639:[1]

"The author regards universal agreement as the criterion of his truths. The norm for my truths, on the other hand, is the natural light of reason. Since all human beings possess the same natural light, one might suppose that they have also the same notions. This is, however, not the case, for almost no one makes good use of this light. And this is why several people can accept the same error. There are many things which could be discovered by natural light, but no one ever thinks of them."

The criterion of truth as established by natural light is thus not to be identified with universal agreement (as it was for Heraclitus and Socrates), although it had its roots in those innate ideas which were called "common notions" in the interpretation given to the Stoic tradition by eclectics and neo-Platonists.

Natural light which Descartes regards as the criterion of truth means accordingly the evidence of conceptual intuition, or the value of clear and distinct ideas. It is true, Democritus and Plato, as we have seen, preceded him on this road, and Galilei also maintained in a certain way the same view. The French philosopher, however, made it his own, thanks to the clearness and force with which he proclaimed it as well as to the consistency with which he used it, especially in his reflections on nature. The purity of rationalism here breaks through certain errors or theoretical prejudices, which may in-

[1] II, 596-7.

deed diminish the value of his physics; but these preju-
dices show all the more how significant his philosophy
is for the history of human thought. His philosophy gains
especially in importance through the moral value which
he bestows upon clearness of consciousness (confusion
and error being the fruit of an obscure will). This has
been expressed by the ancient Skeptics in a beautiful sen-
tence mentioned by Sextus Empiricus: "Those who
speak obscurely resemble people who for some reason
or other shoot arrows in the dark."

The scientific importance of evidence and conceivabil-
ity comes above all to light in the Cartesian mechanics.
Our philosopher goes here back to Parmenides of Elea
(in the sense indicated in the *Timaeus*), rejecting the
void and regarding space as corporeal.[1] It is also here
that the Parmenidean concept of extended matter leads
him to the denial of the external existence of sense quali-
ties,[2] a view maintained already by Democritus and
Galilei. The significance of this criterion is especially
brought out, quite in conformity with a correct interpre-
tation of Eleatic doctrines, in the views concerning the
relativity of motion.[3] (There is nothing more positive in
the motion of a person inside a boat than in the rest of
a second person who sees the first recede from the shore.)
Descartes sums up these views at the end of the second
part of his *Principia Philosophiae* in the following words:
"I have accepted in physics no principles other than those
in geometry or in pure mathematics, for in this way all
natural phenomena can be explained and demonstrated
with certainty." [4]

[1] V, 345; VIII, 49.
[2] VIII, 45-51; cf. V, 190, 238-39.
[3] V, 348; VIII, 48, 53, 57.
[4] VIII, 78.

The value of the Cartesian views can also be found in the fact that they give expression, at least to some extent, to certain demands of reason (understood as an intuitive faculty). It is true that these have been discarded and have yielded place to other hypotheses which correspond better to facts. They, nevertheless, tend naturally to reappear in the history of Science. Thus we see the ether take the place of the void, and we witness to-day a revival of the relativity of motion—in opposition to Democritus, Galilei, and Newton.

12. THE LOGIC OF PASCAL AND OF PORT ROYAL

The ideas of Descartes are reflected in the logic of Blaise Pascal (1623-1662) and in a small treatise coming from the same circles (its authorship is ascribed to Arnauld and Nicole), entitled *Logic or the Art of Thinking* and commonly designated by the name of Port Royal Logic.

In Pascal's *Pensées*, and particularly in articles II and III of the first part of Bossut's edition,[1] an absolutely perfect logical order of science is referred to in the following words:

"This true method would enable one to form demonstrations in the most excellent manner. If it could be realized, it would consist of two main points: . . . to define all terms and to prove all propositions." But "this method . . . is absolutely impossible. For it is obvious that in order to define the first terms it would be necessary to presuppose preceding terms, and similarly, in

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[1] Cf. *Pensées de B. Pascal* (Paris, 1836), pp. 59-90.

order to prove the first propositions it would be again necessary to have preceding ones. In this way it is clear that first terms or propositions could never be reached. By pushing our analysis further and further back we necessarily arrive at primitive words which can be no longer defined and at principles which are so clear that no clearer ones could be found to serve as a basis for proof." Like geometry, this would represent the most perfect logical order that could ever be attained by man. Pascal defines this method by means of the following rules:[1]

Rules for definitions.

1. One shall not try to define things which are so self-evident that no clearer terms could be found for their explanation.

2. Obscure or ambiguous terms shall not be left undefined.

3. Only words perfectly known or already explained shall be employed in the definition of terms.

Rules for axioms.

1. However clear and evident necessary principles may be, we must not employ them unless we explicitly postulate that they be granted.

2. Only perfectly self-evident things shall be laid down as axioms.

Rules for demonstration.

1. One shall not try to prove things which are so self-evident that nothing clearer could be found as a basis for their proof.

2. All propositions that are somewhat obscure shall be proven, and only self-evident axioms or propositions

[1] Ibid., pp. 82-83.

already granted or demonstrated shall be used in these proofs.

3. One shall always substitute in one's mind the notions defined by the definitions in order that one may not be misled by the ambiguity of terms which definitions precisely aim to prevent.

The same rules are to be found in a less precise form in the *Port Royal Logic*.[1] We find here not only a clear expression of the classical requirements put forward by geometricians in regard to their science but also several noteworthy additions:

1. The distinction between axioms and postulates is given up; equal evidence is required now for all primitive propositions.

2. All axioms, however evident they may be, have to be enunciated, and it must be explicitly demanded that they be granted.

3. The preceding demand is coupled with a purely nominalistic conception of definition: "Only what the logicians call *nominal definitions* are admitted in geometry."[2] This constitutes a necessary condition which bestows meaning upon demand 2. We shall examine later in detail the significance of this point as well as the development of the ideas implied by it.

Here we shall only remark that Pascal represents the summit of the Cartesian criticism of logic. His ideas contain implicitly the demands of contemporary criticism, which are not, to be sure, explicitly formulated or even recognized in concrete applications. The example of Frenchmen—like Hoüel—who revived the rules of Pascal in the nineteenth century and used them as guiding

[1] Part IV, ch. X.   [2] *Pensées*, loc. cit., frag. 60.

criteria for the enunciation of the postulates of geometry, shows clearly how insufficient they are for the establishment of a true logical structure of science.

13. Gassendi's and Hobbes' Criticism of the Cartesian Intuition

The whole value and meaning of the logic of Descartes depends on the sense in which his *intuition* is understood, since it is supposed to determine the evidence of principles. This sense is, however, doubtful and obscure. If we consider the use which he practically makes of intuition, for instance, in the construction of his mechanics, it does not seem to differ from imagination or a sense representation that may assume an abstract and simplified form. Descartes himself emphatically rejects such an interpretation. Intuition is for him a vision of purely intellectual concepts, something which, according to the Platonic doctrine of the *Theaetetus*, has nothing to do with imagination or the senses. When our philosopher wants to give us a deeper insight into his views, he shows us thought in the very act in which it immediately grasps the logical relations obtaining between certain terms, seeing in one point all the possible contradictions involved in the ideas before the mind. In such a case intuition seems to be eliminated altogether, its place being taken by definition of terms and logical deduction.

The discussions to which the Cartesian conception of thought was subjected especially by Gassendi and Hobbes reveal clearly the two different interpretations given to it.

Gassendi's criticism is to be found in the fifth objec-

tions to Descartes' *Meditationes de Prima Philosophia*.[1]
Starting his argument from the fifth meditation, our
critic clearly says that all ideas have their origin in sensa-
tion and that there are no inborn ones:

"If you had been deprived of all the functions of the
sense organs so that you had never seen anything or
never touched the surfaces or extremities of bodies, do
you believe that you could have arrived by yourself at
the idea of a triangle or any other figure?"

When he comes to deal with the sixth meditation, he
says, in connection with the chiliagon, that the meaning
and force of the word may enable us to conceive this
figure in some way or other, but they certainly cannot
make us *conceive*, or *imagine* its thousand angles. Con-
ception and imagination, he explains immediately after-
wards, are one and the same faculty; they differ only
in degree.

In the third objections [2] (which belong to Hobbes),
and namely in the fourteenth objection to the fifth medi-
tation, one can furthermore find the following remarks:

"The idea which our mind has of a triangle comes
from another triangle which we have seen or which
we have invented after the model of the things which we
have seen. Once we have given the name of triangle to the
thing from which we believe the idea of the triangle to
have originated, the name will persist even when the
thing ceases to exist."

And in the fourth objection (p. 177):

"There is a great difference between imagining, that
is, having a certain idea, and conceiving with the un-
derstanding, that is, inferring by means of judgment that

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[1] *Œuvres*, VII, 256-391.   [2] Ibid., VII, 171-196.

a certain thing is or exists; but Mr. Descartes does not explain in what the difference consists. . . . ''

"Shall we perhaps say that by reasoning we mean a combination and concatenation of names by means of the copula 'is' (*copulatio et concatenatio nominum sive appellationum per verbum hoc est*)? It would follow from this that by means of reason we can infer nothing in regard to the nature of things but only concerning their denominations; in other words, reason can enable us to see whether we have combined the names of things properly or improperly, in accordance with the agreement which we freely have made about their meanings. If this is the case, reasoning will depend on names, names on the imagination, and the imagination perhaps, in my opinion, on the motion of the bodily organs . . ."

It is obvious that ideas have for Hobbes their origin in sensation. In his attempt, however, to differentiate between the understanding and the imagination, he comes to identify the former with formal logical reasoning, which in its turn is reduced to a pure combination of names established by rules of agreement. We find here a clear expression of a nominalism coupled with sensualism, and in a form strongly suggestive of the tendencies of certain contemporary Italian mathematical logicians.

But in order to understand properly the doctrine of Hobbes, it is necessary to examine his conception of *nominal definition*. And to this end we shall have to retrace our steps a little.

14. Real and Nominal Definitions

Aristotle always regarded definitions as explanations of the essence of things, apt to indicate their causes.[1] The view which sees in definitions only an explanation of names [2] is declared by him to be absurd.

This very repudiation reflects a distinction between real definitions, which tell "what the thing is," and *nominal* definitions. The latter are considered as useless and irrelevant, being similar to the conventions by which the name Peter or John is imposed upon a newborn child. We may add that the belief in the value of nominal definitions repudiated by Aristotle had been upheld against the Platonic doctrine by philosophers like Antisthenes; reference to this can be found in a passage in the *Metaphysics* (VII, 3, 6).

We now are in a position to see to what extent the Aristotelian doctrine was transformed at the hands of a scholastic philosopher of the fourteenth century, like Occam.[3] This thinker, who especially follows the logical tradition transmitted by the Arabs,[4] distinguishes between two kinds of definitions—real and nominal—recognizing explicitly the value of the latter. Let us quote the following passage: "Deffinitionum quædam est deffinitio exprimens quid nominis, et quaedam est deffinitio quid rei. Deffinitio quid rei non est necessaria disputanti scienti significatum vocabuli, . . . quia talis deffinitio non tantum exprimit quid nomen significat, sed exprimit quid

[1] An. Post. II, 9 (7).

[2] An. Post. II, 7 (6).

[3] William Occam, *Summa Totius Logicae* (Oxoniae, 1675) (Book 1, § 26).

[4] Cf. Prantl, op. cit., vol. III, pp. 366, 410.

res est . . . Aliae sunt deffinitiones importantes quid
nominis, quae non sunt nisi orationes exprimentes quid
nomina significant; et tales deffinitiones proprissime sunt
nominum negativorum et connotativorum et respecti-
vorum . . . alia exprimens quid rei non est proprissima
. . . quia tale connotativum non habet nisi deffinitionem
exprimentem quid nominis tantum . . ."

On the other hand, it stands to reason that geometri-
cians, owing to their use of nominal definitions, had a
more adequate conception of the importance of such
definitions. But it is only in Candalla's preface to his
commentary to Euclid [1] that I have been able to find a
clear statement—in opposition to Aristotle's doctrine—
about the nominal character of all definitions:

"Since definitions are only free declarations of names
imposed upon things (*nominum liberae rebus imposi-
tiorum expositiones*), they cannot inform us about the
causes of things . . . ; freedom is not obliged to give
the causes of its actions. Although the imposition is free,
the subsequent use of names is necessary; for otherwise
we should get involved in contradictions. The freedom
of imposition has thus given rise to the necessity of
observance."

Candalla appears in this way as a forerunner of
Pascal, Hobbes, and the modern logicians. But a long
time had to pass before these views found a more gen-
eral acceptance.[2] What above all prevented the recogni-

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[1] Paris, 1566.

[2] Not all logicians, especially non-mathematical logicians, accept them
even to-day. We shall meet in § 16 with a clear statement of the views
of John Stuart Mill. Sigwart also recognizes that: "all logical definitions
are nominal, real definitions being due to a confusion of logical and
metaphysical questions" (*Logik*, 3rd ed., 1904, p. 379). But other au-

tion of the nominal conception of definition was the influence of the Aristotelian prejudice in regard to its value. What apparently is more indifferent and accidental than the fact that this or that name is attributed to a given thing? And how can the choice of one word rather than of another entail important consequences for science?

The value of nominal definitions, nevertheless, gradually gained recognition. The *Port Royal Logic* informs us about the attitude of the Cartesian circles towards such definitions. We find there the distinction between nominal and real definitions; the former are treated in chapters XI, XII, XIII of the first part. The arbitrary introduction of a sign or sound is useful, because it enables us to avoid the ambiguities springing from the confusions of ordinary language. Real definitions are dealt with in chapter XII of the second part. They are said to explain the nature of things by means of essential attributes, in accordance with the Aristotelian view. We are explicitly warned not to confuse the two kinds of definitions, nor to ascribe to the one the arbitrariness belonging to the other. Only Pascal, who rigidly assumes that the criterion of geometry is the logical norm of science, observes, as we have already seen, that we have to recognize only nominal definitions, whose relative character he explains well.

It is interesting to compare the views of these Cartesian logicians with those expressed in regard to the same question in the *Institutio logicae* [1] by another mathematician, John Wallis (1616-1713), the author of the *Arithmetica infinitorum.*

thors, for instance, Ziehen (*Lehrbuch der Logik*, 1920), go back to real definitions.   [1] Oxoniae, 1867.

When the meaning of a word and the nature of a thing are unknown or uncertain—says he [2]—it is necessary (or convenient) to explain and determine the meaning of the word and the nature of the thing by means of words that are more known. This explanation (of the unknown) or determination (of the uncertain) is called *definition,* or when it is less complete—description. Wallis then goes on to distinguish between *nominal* and *real* definitions. Euclid and the mathematical writers were, according to him, the first to use nominal definitions in order to avoid the ambiguities involved in the use of words, phrases, and formulas whose meaning is not well known, or which have already been employed in a different sense. He remarks that naturalists, like mathematicians, employ nominal definitions in order to expound briefly the nature of a thing, and that they call this exposition *real definition.* Mathematicians have a liberty denied to naturalists, because things are what they are and not what we want them to be. As an example of the arbitrary character of mathematical definitions, our author mentions the fact that Euclid applies the name "cone" to the circular right cone, whereas Apollonius understands by this word also the oblique cone, and that "triangle" which is plane for Euclid can also be spheric for Theodosius. We have to point out here that our author did not overlook the importance of conceptual extensions made possible by the widest use of terms.

But what above all determined the use of nominal definitions as an expression of a process of conceptual construction was the introduction of algebraic symbol-

[1] Part I, ch. XXIII.

ism (Vieta), which was, as we have said above, one of the fundamental factors in the evolution of modern logic. The definition of a quantity, for instance:

$$x = (a + b)^2 - a^3 - b^3 + (c - d)^2,$$

brings indeed to light a series of operations which have to be applied in given quantities $(a, \ b, c, d)$ in order to establish x.

The influence of algebraic calculus shows itself clearly in the logical considerations of Thomas Hobbes. We see this in a characteristic manner in the title "Computatio sive Logica" given by Hobbes to the first part of his work *On Body* [1] as well as in the following declarations: "By ratiocination I mean computation . . . Ratiocination is the same with addition and substraction."

In this work Hobbes explains his conception of mathematics, for which he has great admiration. Demonstrative science, he says, is built upon certain primary propositions, but these principles of demonstration are nothing but definitions:

"Now primary propositions are nothing but definitions or parts of definitions, and these only are the principles of demonstration, being truths constituted arbitrarily by the inventors of speech and therefore not to be demonstrated" (Ch. III, 9).

Hobbes does not agree with those thinkers who "have added to these propositions others, which they call primary principles, namely axioms and common notions. For though they be so evident that they need no proof, yet because they may be proved are not truly princi-

[1] *Elements of Philosophy Concerning Body.* The English Works of Thomas Hobbes, edited by Sir William Molesworth. Vol. I. London.

ples." And as to certain petitions or postulates, as, for example, that "a straight line may be drawn between two points," these are indeed "the principles of art or construction but not of science or demonstration."

These criteria are taken up in ch. VI On Method, where the properties of definition are given (no. 15):

1) It takes away equivocations;

2) It represents a certain universal picture of the thing defined not to the eye but to the mind;

3) It is not subject to discussion whether it is to be admitted or not;

4) It precedes defined terms in philosophy, that is, compounded terms, the compositive process of which it represents;

5) It can be changed arbitrarily when we pass from one part of science to another;

6) It must not repeat the defined term. For a defined term is the whole compound, and definition is the resolution of that compound into parts, but no total can be part of itself.

We have quoted these rules in order to show our author's conception of definition, in which he evidently was guided by the definition of a quantity by means of a formula. Let us now see to what criticism these views of Hobbes were subjected by Leibniz.

15. The Rationalism of Leibniz and the Criticism of the Logically Possible

The great German philosopher Gottfried Wilhelm Leibniz (1646-1716), who shares with Newton the glory

of the discovery of the calculus (or of its systematiza-
tion), represents the highest expression of the mathe-
matical type of the metaphysical rationalism which domi-
nated the most eminent minds of that fruitful period.
The eclectic mind of Leibniz exhibits in reality various
aspects tending to combine the new scientific views with
the scholastic tradition. The scholastic part shows itself
most clearly in a metaphysical conception derived from
an analysis of propositions, as has been discovered by
recent critics and interpreters of the *Monadology*.[1] The
reflections on the principles of science, upon which we
want to throw some light here, are, however, by far more
significant.

For Leibniz, just as for Parmenides, Democritus, and
Descartes, the criterion of truth is to be found in thought.
It is true that a new meaning is acquired now by the
Parmenidean rationalistic motive (the object of thought
and the existent are the same), which functions in the
system of Democritus as an assumption that every ra-
tionally conceivable thing must be realized somewhere in
the infinite universe. The geometric model of science, to
which Descartes still clings, becomes replaced now by a
mechanical one. As a result of this the concept repre-
sents for Leibniz no longer the criterion of existence but
of *possibility*. Another principle, that of *sufficient rea-
son*, is made to determine the transition from possibility
to actuality and *existence*.

It is to this end that he establishes the following dis-

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[1] Among these are to be counted the mathematical logicians Louis
Couturat, the editor of an important series of fragments to which we
shall soon refer, and Bertrand Russell, *A Critical Exposition of the
Philosophy of Leibniz* (Cambridge, 1900) ; we shall speak about the lat-
ter work in § 28.

tinction: *Ens quod distincte concipi potest. Existens quod distincte percipi potest.*[1]

Now this distinction between possibility and existence, which is connected with our philosopher's mechanistic conception of the universe, virtually contains an emancipation if not of metaphysics at least of logic from the realism of Aristotle in so far as the concept is regarded now as a product of mental activity. The historical development of the Leibnizian ideas would show that this is the true meaning of the wider reality assigned by our philosopher to the mathematical sciences in his attempt to extend them beyond the field of physical application. But in order to have a more exact understanding of this conception, it would be necessary to find out the meaning he attached to "conceivability," which became for him a criterion for determining the principles of demonstrative science, and accordingly of geometry.

He objects to the Cartesians and especially to Pascal's rules on the ground that the criterion of the evidence of principles is fallacious as long as we are not given "quelques marques" enabling us to recognize what is obscure and doubtful. And he adds: [2] " . . . I am convinced that in order to make the sciences perfect it would be necessary to prove certain propositions which are called axioms. We should have to follow in this respect the example of Apollonius who tried to demonstrate some of the principles which Euclid assumed without proof." And after mentioning a similar attempt on the part of Roberval, our author remarks that, while the demonstration of axioms is not necessary for beginners,

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[1] Cf. *Opuscules et fragments inédites de Leibniz,* par L. Couturat (Paris, 1903), p. 437.   [2] Op. cit., p. 181.

nothing is, however, more necessary for those who want to pass the Herculean pillars of knowledge.

Leibniz comes back to these attempts of Apollonius and Roberval in another fragment entitled "Demonstratio Axiomatum Euclidis." [1] He praises these mathematicians, for we can arrive at perfect knowledge by demanding nothing of the senses and the imagination and by expecting everything from concepts (notiones). Analysis of concepts has indeed to lead to purely logical or identical primary truths, from which the demonstration of all their properties is to be deduced, except in the case where the analytic process involves an infinite regress.

This view is clearly expressed in an article called "Primae Veritates" (op. cit., p. 518): "Primae veritates sunt quae idem se ipso enuntiant aut oppositum de ipso opposito negant. Ut A est A, vel A non est non A. Si verum est A esse B, falsum est A non esse B vel A esse non B. Item unumquodque est quale est . . . aliaque id genus, quae licet suos ipsa gradus habeant prioritatis, omnia tamen uno nomine *identicorum* comprehendi possunt.

"Omnes autem reliquae veritates reducuntur ad primas ope definitionum, seu per resolutionem notionum, in qua consistit *probatio a priori,* independens ab experimento."

The example which follows (namely, the demonstration that "the whole is larger than—and not equal to—its part") serves to elucidate his thought, although the argumentation is not quite satisfactory.

John Bernouilli had objected in a letter of August 15, 1696, to the demonstration of axioms and maintained that it was necessary to have some indemonstrable ax-

[1] Op. cit., p. 539.

ioms by means of which theorems could be proven. To this Leibniz replies that the only indemonstrable axioms are identical propositions; he insists also on several other occasions on this idea, which forms one of the fundamental motives of his speculation. He says, for instance, in a fragment for a new encyclopaedia [1] written January, 1679, that the propositions of all sciences are either principles or conclusions. Principles are either definitions or axioms . . . ; definitions are arbitrary (they have of course to conform to common usage) . . . ; axioms are those propositions which are regarded by every one as evident and which prove upon closer examination to follow from the definitions.

All this might suggest that the conception of Leibniz does not differ essentially from that of Hobbes, and that the mathematical philosopher only gave the latter's views a more precise determination. A more attentive study, however, of the Leibnizian ideas taken in their mature and precise form shows that he was aware of the "difficultas hobbesiana de veritate arbitraria" and that he tried to meet it.

Vailati justly remarks in his essay "L'influenza della matematica sulla teoria della conoscenza nella filisofia moderna" [2] that a philosopher like Leibniz could not commit the error of regarding mathematical definitions as *entirely* arbitrary. For he knew that since the days of the Greek geometricians it was always required to accompany the definition of a figure with a proof of its existence. The Greeks accomplished this by means of *constructions* based on first *postulates*. Owing to the recognition of this scientific function of postulates, Leibniz

[1] Op. cit., p. 32.   [2] *Scritti*, pp. 599-616.

could not be satisfied with the view of Hobbes according to which such principles belong to art rather than to science. Leibniz explicitly mentions the case of the *regular decahedron,* which is an impossible figure. He remarks that even if we took the liberty of introducing it into geometry by definition, we would only make evident, by working out its implications, the contradictions implicitly contained in its concept.

This leads our philosopher to determine the concept of the *possible* [1] and hence to divide definitions into *real* and *nominal.* He calls real any definition which sets forth the *real decomposition of a logically possible concept into simple concepts,* and it is thanks to this decomposition that he proves the logical possibility of concepts (that is to say, their correspondence to entities or to subsistents). This demonstration is always assumed as possible, but in the case of its absence the definition is called real when it is accompanied in some way or other by the proof or postulate of the logical existence of the concept, that is, by the statement that its attributes are not contradictory.[2] A definition, on the other hand, is purely nominal if it indicates the distinguishing traits of a thing without showing that the thing is possible.

The Leibnizian theory of definition is to be found in his essay "Meditationes de cognitione, veritate atque ideis," to which many of the fragments published by Couturat refer. Its starting point is here the so-called *combinatory art* or *general characteristic,* that is, the uni-

[1] "Possibiles sunt termini de quibus demonstrari potest numquam in resolutionem occursuram contradictionem" (op. cit., p. 371).

[2] Definitio realis seu definitio talis ex qua statim patet rem de qua agitur esse possibilem (op. cit., p. 220).

versal symbolism, which was regarded by Leibniz as a means for extending the language of algebra to all rational sciences. It is this that gave later rise to *symbolic* or *mathematical logic*. This Leibnizian idea was anticipated (apart from the attempt of a late Epicurean, Philodemus, in antiquity) by the *Ars magna* of Raymond Lully (1234-1315), the *Ars signorum* of Dalagranus (1661), and the *Essay towards the Real Character* of Wilkins [1] (1668).

It is thus clear that Leibniz, much more than Hobbes, regards definitions as being similar to an expression of a quantity by means of an algebraic formula. Hence the new rule which Leibniz substituted for the scholastic rule. Definitions, he said, should include the necessary and sufficient conditions for the demonstration of the properties of the notion defined: "Eiusdem definiti multae possunt esse definitiones," for "omnis proprietas reciproca potest esse definitio." [2]

The algebraic expression of x by means of the quantities a, b, c . . . cannot, however, truly define x, unless the symbols figuring in this expression represent possible operations (division by zero, for instance, must be excluded), and unless the elements a, b, c . . . to which the operations are applied are known quantities, or are defined in their turn in terms of elements that are already known. In order, therefore, to satisfy the conditions for real definitions, according to which: "In omni definitione constare debet id quod definitur esse possible," [3] it is necessary that the construction of the concept given

[1] Cf. L. Couturat, *La logique de Leibniz* (Paris, 1901).

[2] "Essai de calcul logique" in *Opuscules,* op. cit., p. 258.

[3] Op. cit., p. 328.

by the definition should be capable of decomposing it into elements whose possibility is immediately evident. This occurs in the case of simple ideas, which cannot be, in his opinion, contradictory,[1] and of which he even planned to give a catalogue.[2] Couturat [3] points out, on the basis of the published manuscripts, that Leibniz always conceives the decomposition or analysis of a complex concept into simple ones after the fashion of the decomposition of a whole number into its prime factors; this analogy suggests to him the possibility and the uniqueness of such an analysis.

What value shall we bestow upon the Leibnizian theory of definition? In the first place we can say that the analysis contained in it removes from logic the Aristotelian conception of real definition (the definition which simply indicates an object given outside us). Although this view seems to show signs of life here and there, we can say with Ariosto:

> Il poverin che non s'era accorto
> andava combattendo ed era morto.

In the second place we may remark that this analysis raises in the history of science the problem of *the compatibility of a system of propositions* which are attributed to a concept as marks. In other words, we are here confronted with the question whether given premises which may not be explicitly contradictory, may, never-

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[1] "Meditationes de cognitione, veritate et ideis" in Dutens *Leibnitii opera omnia* (Genevae, 1768), vol. II. Cf. *Opuscules,* etc., pp. 195 and 219, note 21.

[2] "Catalogus notionum primarium ex quibus ceterae pleraeque omnes componuntur," in op. cit., p. 400.

[3] *La logique de Leibniz,* p. 192 ff.

theless, involve an implicit contradiction; and if this is the case, we especially have on our hands the problem of how to remove this doubt.

We have also to point out that the Leibnizian answer to this problem, namely the unique decomposition of complex concepts into simple ideas, is connected with his realistic presupposition and with his characteristic attitude towards deductive definition (from the general to the particular). The Leibnizian logic as a rule considers concepts intensionally (that is, as a combination of the most general concepts corresponding to their attributes) rather than extensionally (that is, as classes of the represented objects).[1] The definition of a concept is thus made to depend upon the highest genera, which must be simple concepts in order to be truly the highest and not subordinate to more universal ones.

Such a procedure does not seem to make the Leibnizian view very fruitful. This seems to be also the opinion of Vailati,[2] who brought to light the importance of the ideas of our philosopher as a result of deep studies. We shall, however, see later that a more precise meaning can be given to the Leibnizian conception by transposing it from intensional to extensional logic. This is essentially the significance of the work started by us in 1902,[3]

[1] This is, for instance, seen in the notation $A + B$ by which he designates the concept comprising both the attributes of A and of B. (Cf. "Non inelegans specimen demonstrandi in abstractis," ed. Erdmann, p. 94.)

[2] Loc. cit., 1905, cf. *Scritti,* p. 617. He praises Leibniz for having attempted to give an empirical proof of the compatibility of postulates, which naturally must have been regarded by the philosopher himself as a makeshift and which in any case is far from solving the difficulty (cf. § 28).

[3] Expounded in lectures given at the universities of Bologna and Brussels and published afterwards in ch. III of the *Problemi della*

a period in which we knew nothing of the views of Leibniz. (Cf. § 28.)

16. The Theory of Definition in Saccheri

Closely related to the ideas of Leibniz seem to be those expressed by Gerolamo Saccheri (1667-1733) in a long forgotten booklet called *Logica Demonstrativa*.[1] It was thanks to Vailati that it was rescued from oblivion. The coincidence in the ideas of the two thinkers is all the more remarkable as Saccheri seems to have taken his starting point not from the symbolism of algebraic calculus but from a geometrical question, namely from the famous fifth postulate of Euclid, the basis of the theory of parallels. The various attempts made by many thinkers to demonstrate this postulate must have occupied his mind long before he offered the world his own reflections in a work called *Euclides, ab omni naevo vindicatus* and published in Milan in the year of his death. According to Beltrami, this book represents the most important anticipation of the non-Euclidean geometries of Lobatchevsky and Bolyai.

Of all the attempts to prove this postulate those of Posidonius and Geminus in Proclus, renewed by Borelli (1658), must especially have attracted the attention of Saccheri. For they offer a theory of parallels deduced, without the help of a new postulate, from the definition of parallels as co-planar, equidistant straight lines. Of course, it seems rather strange to believe that the diffi-

Scienza, translated into English by Katharine Royce under the title of *Problems of Science* (Chicago, 1914).

[1] Published for the first time in Turin in 1697. I have found a copy of the second ed. (Pavia, 1701) in the library Vittorio Emanuele of Rome.

culty of proving the postulate could be removed by the mere change of definition.

Saccheri (who was preceded by the criticism of Giordano Vitale da Bitonto) realized quite well that this could not be done *sine magno in logicam peccato*. For the complex definition which represents parallels as equidistant straight lines presupposes implicitly that the locus of points equidistant from a straight line is in its turn a straight line. The theory of Saccheri makes evident the fallacy involved in this reasoning, and even if it was not constructed for this purpose, it seems to have been at least greatly inspired by this example.

Saccheri apparently tried to adhere closely to Aristotle. He even reproduced the divisions of the *Organon* (Part I, Analytics; Part II, Posterior Analytics, etc.). Especially, the third chapter of the second part (op. cit., p. 116) begins with the Aristotelian classification of principles: definitions, axioms or assumptions, and postulates. After pointing out that no science can prove its whole object and that it has partly to postulate it, he passes to a distinction between *real* and *nominal definitions*. He says in regard to the latter that: "explicat vocis significatum" and "nata est suadere definitio quid rei per postulatum, vel dum venitur ad questionem *an est* et respondentur affirmative (p. 118).[1]

Real definition thus reduces itself for Saccheri to nominal definition to which a postulate or demonstration of existence is added. It is evident that his conception dif-

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[1] As a curiosity we may cite the example which he gives in this connection of the polemic of Zeno of Elea against the generation of a continuum from "points."

fers only verbally from that expressed so clearly by John Stuart Mill.[1]

"The distinction between nominal and real definitions, between definitions of words and what are called definitions of things, though conformable to the ideas of most of the Aristotelian logicians, cannot, as it appears to me, be maintained. All definitions are of names, and of names only: but, in some definitions, it is clearly apparent that nothing is intended except to explain the meaning of the word; while in others it is intended to be implied that there exists a thing corresponding to the word. Whether this is or is not implied in any given case, cannot be collected from the mere form of the expression. There are, therefore, expressions, commonly passing for definitions, which include in themselves more than the mere explanation of the meaning of a term. But it is not correct to call an expression of this kind a peculiar sort of definition. Its difference from the other kind consists in this, that it is not a definition, but a definition and something more."

Saccheri arrives subsequently at a deeper conception of definition. He comes to the conclusion that real definition (quidditativa) is not a necessary principle of science, because it can present itself as a scientific conclusion, whereas nominal definition has to precede every other concept of the thing denoted by the term (p. 123). Every nominal definition is good and can become only an object of historical controversy (p. 126). But when we are concerned with the existence of the thing defined, we must ask whether a part of the definition does not by chance indicate essential properties which are suffi-

[1] *System of Logic* (1st ed., 1843), 6th ed., VIII, 5.

ciently determined by another part. In the case of such complex definitions it is not easy to admit the postulate; its admission, however, does not present any difficulty whenever we deal with non-complex definitions (incomplexae).[1]

These ideas, which are again taken up in the Sophistic (part IV) in connection with the fallacies of reasoning, betray a close relationship with those of Leibniz. We miss, however, in Saccheri the conception of definition as a constructive mental process, which was suggested to Leibniz by the symbolism of the calculus. From this point of view his logic still remains under the influence of ancient realism, although he often seems to interpret scholastic expressions in the same sense as was shown in the analysis of his contemporary. Thus, we hear him say that, in contradistinction to the axiomatic, immediately certain propositions of which the principle of contradiction is an example, there are other axioms which cannot be reduced to simple analytic or identical propositions and which are immediately evident solely through the meaning of their terms (ex sola terminorum intellectione), for instance, that the whole is larger than any of its parts.[2] He seems to appeal here to that imaginative vision of things reached through names which Leibniz wanted to exclude, despite his frank attempt to prove this axiom.

17. The Psychological Criticism of Locke

The history of seventeenth-century thought offers the student an extremely attractive spectacle. Thinkers had

[1] Op. cit., pp. 129-30.   [2] Ibid., p. 127.

then not only a clear understanding of tendencies differing from their own but also a common desire to collaborate in the solution of the same problems, a fact which bestows a marvelous unity upon the work of the philosophers. We have already pointed out the intimate connection between the logical views of Descartes, Hobbes, and Leibniz. We shall now turn to another critic of the Cartesian philosophy, a countryman and—to a certain extent—an heir of Hobbes (and also of Gassendi), who represents empiricism in the highest and most conscious manner in opposition to the rationalism of Leibniz. It will be interesting to throw some light on the close similarities which exist between the thoughts of the two eminent adversaries.

John Locke (1632-1704) has expressed the results of his meditations in a work called *An Essay Concerning Human Understanding* (1690), which forms one of the most conspicuous monuments of the history of modern philosophy, despite certain defects and errors that may be ascribed to it. This criticism of the human understanding takes its starting point from a refutation of the Cartesian *inborn ideas*—a renewed form of the doctrine of reminiscence of Plato's *Meno*. The term "idea" generally designates for Locke whatever is an object of thought. After devoting the first book to this refutation, he sets out in the second book to show that all our ideas have their origin in sensation or in reflection. To be more precise, he believes that there are *simple ideas* (II, 2), which are distinct in sensation or reflection and which are not further distinguishable (for instance, coldness, hardness, whiteness, etc.). These are received by the mind largely in a passive manner, as

data. *Complex ideas,* on the other hand, are made by the mind out of simple ones (II, 12).

Simple ideas, which come from sensations, are regarded as "natural and regular productions of things outside us, representing to us things under those appearances which they are fitted to produce in us" (IV, 4, § 4). Locke, nevertheless, admits that it is legitimate to extend thought to what is not directly perceivable, to what is given to us by perception in a constant manner. It is in this sense that he revives the distinction made by Democritus and Galilei between two kinds of qualities of matter. He recognizes as *primary qualities* (II, 8, § 9) those "which are utterly inseparable from body, which sense constantly finds in every particle of matter that has bulk enough to be perceived, and which mind finds inseparable from every particle of matter, though less to make itself singly be perceived by our senses." Such are for Locke not only the Cartesian qualities, *extension and figure,* but also solidity and motion. And these primary or original qualities produce in us simple ideas which are similar to the qualities of the bodies themselves (§ 15).

Secondary qualities, on the other hand (color, odor, etc.), are nothing in the objects themselves but powers to produce various sensations in us by their primary qualities (§ 10), calling forth in us ideas by the operations of insensible particles on our senses (§ 13).

Book III contains a deep examination, from the point of view of nominalism and terminism, of the formation of general ideas. A general term is nothing but an abstraction from a group of ideas associated by the human mind; species are thus the work of the human under-

standing in so far as they are based upon a real resemblance of individual things (III, 3). This study leads Locke directly to the analysis of language. Words are sensible signs used by human beings for communicating their thoughts; they are signs of ideas, although it often happens that we use words which correspond to no idea; as such they are of course meaningless (III, 2).

This brings us to Locke's theory of definition (III, 4, § 6). "Definition is nothing else but the showing of the meaning of one word by several others not synonymous terms." He explains then that to understand the sense of words means to understand "the ideas for which they are made to stand by him that uses them." "The meaning of any term is shown, or the word is defined, when by other words the idea is made the sign of, and annexed to, in the mind of the speaker, is, as it were, represented or set before the view of another, and thus its signification ascertained." But since the different terms of a definition signify ideas, simple ideas cannot be defined (§ 7). The only way of explaining simple ideas is by showing directly the things which produce them in the mind (11, § 14).

At first sight one might think that there is a great difference between the views of Locke and those of Leibniz. Definition seems to be for Locke an appeal to an imaginative vision, whereas Leibniz conceives it as an explanation of logical relations or of the constructive process by which simple ideas are combined. But psychological reality as a matter of fact embraces both aspects, and the second aspect (which we regard as properly logical) does not escape the Lockean analysis. The difference is due to the concrete spirit of the English philos-

opher and to the fact that he describes a mental process, whereas Leibniz has in mind a combination of signs after the fashion of algebra. But even this consideration seems to find a place in Locke's mind, since in book IV he insists on the importance of giving fixed names to clear and distinct ideas (12, § 14), and he mentions the method of algebra as an example of scientific method, foretelling its extension to other fields of our knowledge (§ 15). Finally in chapter 21 of the same book, where he deals with the classification of the sciences (§ 4), he speaks of a doctrine of signs (σημειωτική) as a means for advancing logic (λογική from λόγος, which means "word"). The object of this science would be to study systematically the nature and function of signs, in this way offering a remedy for the imperfections and misuses of language and a help to exact reasoning.

All this reveals a great similarity to the Leibnizian ideas. The opposition between the two tendencies, on the other hand, shows itself clearly in the opinion expressed by Locke in regard to the analytico-deductive method which claims to deduce science from evident axioms. Such principles are for our philosopher useless and futile, especially when our ideas are determined in our mind and are designated by fixed and known names (IV, 7). Locke expressly alludes to Leibniz in the following words (IV, 8, § 3): "I know there are some who, because identical propositions are self-evident, show a great concern for them, and think they do great service to philosophy by crying them up, as if in them was contained all knowledge, and the understanding were led into all truth by them only. I grant as forwardly as any one that they are all true and self-evident. I grant further

that the foundation of all our knowledge lies in the faculty of perceiving the same idea to be the same and of discerning it from those that are different. But how that vindicates the making use of identical propositions, for the improvement of knowledge, from the imputation of trifling, I do not see. . . ."

He adds that the true method of science, and of mathematics, is something quite different. It consists in the discovery of intermediary ideas which have to be arranged so that the understanding may see the agreement or disagreement of those in question. It is through this, he says in another place, that Newton arrived at his remarkable discovery.

Good English common sense speaks through the mouth of Locke. There remains, however, in the Leibnizian ideas one question which our philosopher has hardly solved by brushing it aside. We may grant that there are no innate ideas and that it is impossible to construct physical reality a priori. The analytic resolution of concepts may, nevertheless, have a value, at least as far as mathematics is concerned. It is legitimate to ask whether we do not find among the concepts constructed by our mind one concept, namely number, which exactly reflects the faculty by which the objects of thought are identified, distinguished, or combined independently of their sense content. And do we not see, on the other hand, how different kinds of real relations are represented in thought in an analogous manner by means of systems of concepts constructed out of elements which become equivalent when we abstract from their qualitative differences?

Even in regard to these ideal constructions Locke might maintain that the art of research would derive little

help from an adherence to identical propositions and that one does not become an algebraist through repeated logical exercises. But the question acquires a different meaning for one who sees in logic itself a *science* having for its object the analysis of thought. Is it possible to construct or reconstruct a mathematical theory as a deductive system without assuming any hypotheses in regard to reality apart from the general existence of objects of thought as they are recognized, distinguished, combined by thought? And can these laws of thought guarantee a priori that no contradiction will arise in the series of successive deductions? Whatever one may think of this question it was not only not solved, it was not even examined by Locke.

18. The System of Newton and the Decay of Metaphysical Rationalism

The Lockean refutation of inborn ideas was not accepted by Leibniz. The arguments of the English philosopher were combatted by him in his *Réflexions sur l'essai de l'entendement humain de Locke* and in his *Nouveaux essais sur l'entendement humain,* written apparently in 1704 but not published, on account of the death of Locke, until many years later in 1765.

Leibniz insists that there are inborn *truths,* for instance, those of arithmetic and geometry, although it is correct to say that in the absence of sight and touch mind would never be led to deal with the corresponding ideas. This does not mean that there exist inborn *thoughts:* thought is an activity, truth only requires attention to be brought to light. Mind has the *disposition*

to find in itself *necessary truths,* which do not come from the senses. It discovers them, so to speak, just as the sculptor discovers preëxisting veins in marble. It is in this way that we acquire the truths of numbers, which are also in ourselves.

In short, to the Lockean aphorism "Nihil est in intellectu quod non prius fuerit in sensu," Leibniz replies, "nisi intellectus ipse."

The real value of the Leibnizian arguments does not seem, however, to go beyond what Locke himself admits when he says that "innateness does not differ from the capacity of knowing and thinking." There is evidently a jump from the admission that the human mind is conscious of the activity of its own thoughts to the statement that it can draw from inner experience notions like being, substance, action, as Leibniz does in his "Meditationes de cognitione, veritate et ideis." [1]

On the other hand, Locke realized well that the great progress achieved by Newton with his theory of universal gravitation was leading to a systematization of mechanics quite different from the one which Leibniz tried in vain to construct. Science *made* does not correspond to the model which was imposed upon a *science to be made.* For the Newtonian system does not confine its starting point to clear concepts and axioms that are a priori evident in accordance with Cartesian or Leibnizian criteria. It rather follows the views of the school of Galilei (Baliani) and introduces additional postulates derived from observations or experiments.

Newton, moreover, transcended the naïve conception of the experimental method. Having deduced the calcu-

[1] *Phil. Schriften,* ed. Gerhardt. Vol. 4, pp. 452-53.

lation of central forces from the laws of Kepler, which were the result of long astronomical observations and of a wise induction,[1] he *generalized the hypotheses* obtained in this way, by assuming the *universality* of attraction between material masses. As a result of this he succeeded in correcting the same Keplerian laws.

Deduction brought to full light appears thus as a means for correcting hypothetical premises, that is, as a proper instrument of induction. The new *conception of science as historical* removes the stumbling block of the ancients: the dilemma between the impossibility of demonstrative science and the necessity of indubitable principles. Science is regarded now as cumulative, as a progress of systems or deductive theories gradually coming nearer to reality, each growing out of the preceding and erecting its consequences into more and more general principles.

The Physicists (like the philosophers) were, however, slow to see this historical conception of science in the lesson suggested by Newton. The enormous success of the doctrine also contributed to the general conviction that rigorous and immutable principles of nature have been discovered here. But the work of those who endeavored to develop the consequences of these principles (which were extended far beyond the field of astronomy) in all branches of physics, practically proceeded in the direction which criticism has to assign to-day to the Newtonian system. Although the objections of the Cartesians and Leibnizians were silenced, they remained as a testimony to certain intellectual demands, which were

[1] Cf. E. Daniele, "I moti planetari e le leggi di Keplero" in *Periodico di Matematiche,* Bologna, July, 1921.

to bring about a renovating crisis at a more opportune
moment (when the Newtonian doctrine will no longer be
able to satisfy the too great hopes called forth by it).

If the rationalistic metaphysics of Leibniz was de-
feated by the Newtonian theory in the field of physics,
it had also to succumb to the advance of the empiricistic
criticism, carried on by the successors of Locke.

Berkeley (1685-1753) was led in his *New Theory of
Vision* (1709) and later in his *Principles of Human
Knowledge*" (1710) to do away with the distinction be-
tween primary and secondary qualities of matter and to
show that the former, like the latter, expressed
nothing but *simple relations between possible sensations*.
There is no justification, therefore, for the belief that the
true intelligible substratum, the *substance* underlying
phenomena, can be found in the evidence of geometrical
or mechanical concepts.

David Hume (1711-1766) pushes this criticism still
further.[1] He attacks first the idea of *substance,* then
that of *cause*. It is erroneous to see in causality a neces-
sary relation by which the understanding connects two
concepts in a rational manner. Causality in the end re-
duces itself to a *constant succession* of contiguous phe-
nomena, to which there corresponds in the mind a con-
stant association of ideas established by habit.

With Hume English psychological criticism reaches its
summit, which is also the summit of human thought. And
if the conclusions of this criticism may appear as *skep-*

[1] Cf. especially his first and most comprehensive work *Treatise of
Human Nature* (1739) and section I of *The Inquiry,* etc.

tical from the point of view of a metaphysical ideal, and if they contain an unsolved problem, they, nevertheless, offer a *positive* affirmation of science and represent an anticipation of that philosophical tendency which was developed in the nineteenth century under the name of positivism. It is Hume who is followed directly as a master by those representatives of this tendency who know best the history of science, by thinkers like John Stuart Mill and Mach, who advocate decidedly empiricistic views. These thinkers refuse to accept the conventional opinion of the parrots of philosophy [1] that the Humean position has been absorbed by the criticism of Kant, or that it has been superseded by any one.

19.   THE LOGIC OF KANT

While the doom of metaphysical rationalism was imminent, several attempts were being made to make the linguistic schemes of logic more perfect by means of a symbolic analysis. The ideal of the Leibnizian "characteristica universalis," of which we have only a few references, largely unpublished, is pursued by the following authors:

J. A. Segner, *Specimen logicae universaliter demonstatae* (1740).

[1] The more zealous ones go as far as to dismiss from the history of philosophy the deep analysis of the "Treatise of Human Nature" under the specious pretext that Kant knew only those doctrines of Hume which are reproduced and popularized in his later essays, the rest being, therefore, of little importance for the progress of ideas. As if the thought of a philosopher representing, in an historical form, a stage of human mind "sub specie aeternitatis" could have no life outside the influence which it may accidentally have upon the work of the nearest thinkers, or outside the conditions which at the moment seem to secure a world-wide success!

G. Ploucquet *Fundamenta Philosophiae speculativae* (1759), etc.

J. H. Lambert, *Neues Organon* (1764). *Anlage zur Architektonik oder Theorie des Einfachen und des Erstens in der philosophischen und mathematischen Erkenntnis* (Riga, 1771).

G. I. Holland, *Abhandlung über die Mathematik, die allgemeine Zeichenkunst und die Verschiedenheit der Rechnungsarten* (1764), etc.[1]

We shall give a brief sketch of the proper significance of this analysis in § 28. We may point out here that the attempt of Lambert (a well known mathematician and one of the forerunners of the non-Euclidean geometry) is especially interesting, as far as we can see, from the point of view of epistemology. Following Leibniz, Lambert is concerned with finding simple concepts as the constitutive elements of knowledge; these he wants to represent by means of signs of a mathematical type. His *Grundlehre* assumes it as a principle that such concepts must exist, for "otherwise there would be no end to definition or demonstration."[2] But Lambert conceives definition intensionally as starting from the most general predicables which form the attributes of the concepts defined; a real significance is to be ascribed to these highest genera. Our philosopher believes that all

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[1] Of all these works I have been able to see only Lambert's *Anlage zur Architektonik*. Reference to the symbolic systems of these authors (with an ample bibliography) can be found in J. Venn's *Symbolic Logic* (1881), 2nd ed. (London, 1894). In the *Revista Mathematica*, edited by Peano, vol. IV, p. 120 (1894), there is also a note on L. Richer's "Algebrae philosophicae in usum artis inveniendi, specimen primum." *Miscellanea Taurinensis*, II, 1761.

[2] *Architektonik*, p. 19.

difficulties will be removed if we cease to use *axioms* furnished by the thing itself and resort instead to *principles* reflecting not the *matter* (Stoff) but the *form* of knowledge so that we should be left with relational concepts only. But since by simple relations we cannot determine things, we should be confined in this way to appearances, and the human "Grundlehre" would and could contain only principles of phenomena and would have accordingly to adapt its theory to practice.[1] We find here interesting ideas capable of throwing light upon the evolution of ideas that led to Kant's philosophy (and also of illuminating the modern pragmatic motives of the mathematical logic of Peirce).

We must not believe, however, that the ideal of symbolic logic springs solely from the realistic tendencies inspired by the philosophy of Leibniz and—going back to his forerunners—perhaps by Raymond Lully's alchemic conception of science. As a matter of fact, we have seen how the empiricist Locke was interested in a doctrine of signs. And it is from him that this ideal passed over to Condillac [2] (1715-1780), who professed a more radical sensualism. The analysis of ideas or the "art de penser" reduces itself for Condillac to "une langue bien faite" (XV, p. 400), since the errors in our judgments are chiefly due to our habit of using words whose meanings we do not analyze. Languages are according to our philosopher imperfect instruments of an inborn "language d'action" (ibid. and XVI, p. 4), algebra being the simplest (XV, 435) and perhaps the only perfect language

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[1] *Architektonik,* p. 39.

[2] *Œuvres complètes,* vol. XV (Paris, 1827); *La logique ou les premiers devéloppements de l'art de penser* and vol. XVI. *Langue des Calculs.*

(XVI, 5); he studies it as a method for the analysis of thinking.

The influence of these views showed itself only later in the development of logic in the nineteenth century. Immanuel Kant, who was a friend of Lambert and admired greatly this remarkable man, was quite preoccupied with the skeptical consequences which the defeat of rationalism seemed to involve. Unlike the empiricists who abandoned the Aristotelian or Leibnizian realism, he could not satisfy himself with a psychological conception. For in the reality of thought, as expressed in words and signs, these thinkers saw an empirical fact only; they confined themselves to an investigation of the origin of its most elementary forms as found in children and savages and in this way they were unable to distinguish what is of real value for science. Kant, on the other hand, starts from science made, in order to get back by a regressive method to the principles which make science possible. It is for him an incontestable fact that mathematics (and also pure physics which includes rational mechanics) has a necessary character which cannot come from experience. But at the same time he takes the part of Locke against Leibniz and maintains that the principles of these sciences cannot be reduced to identical and analytical judgments.

The rationalism involved in the methods of Galilei, Torricelli, and the chemist Stahl is heralded by Kant in the preface to the second edition of the *Critique of Pure Reason* [1] in the following words:

[1] *Kritik der reinen Vernunft,* 1781, 2nd ed. The English translation is that of F. Max Mueller, *Kant's Critique of Pure Reason* (London, 1881), vol. II, p. 368.—Note of the translator.

"They comprehended that reason has insight into that only which she herself produces on her own plan, and that she must move forward with the principles of her judgments, according to fixed laws, and compel nature to ask her questions, but not let herself be led by nature, as if it were in leading strings, because otherwise, accidental observations, made on no previously fixed plan, will never converge towards a necessary law, which is the only thing that reason seeks and requires. Reason, holding in one hand its privileges, according to which concordant phenomena can alone be admitted as laws of nature, and in the other hand the experiment which it has devised according to those principles, must approach nature in order to be taught by it: but not in the character of a pupil, who agrees to everything the master likes, but as an appointed judge, who compels the witnesses to answer the questions which he himself proposes."

This experimental, rationally conducted enterprise rests, however, for Kant on synthetic a priori principles, which are non-identical judgments and in which the data of experience are accompanied by a necessary order produced by mind. The analysis of these principles presupposes the distinction between the *form* and *matter* of knowledge; the matter being the external datum, the form reflecting the nature and function of the thinking organ. To be more precise, the forms of sensibility are the intuitions of space and time, from which the axioms of geometry and mechanics are derived, while the forms of the understanding supply the logical axioms.

These a priori forms are for Kant something which thought derives from its own depths. They are conceived

by him in a rigidly determined manner, so characteristic of his conception of science—of the Euclidean geometry and Newtonian mechanics—, which is supposed to be a definitive and immutable acquisition. Moreover, he does not give us a proper criticism which starts from science made in order to go back to the conditions and meaning of knowledge, as was developed later—although in a different sense—by Auguste Comte. His justification *de jure* goes so far as to prescribe boldly the principles which have to serve as a basis for the extension of science. But this attempt, made in the *Metaphysische Anfangsgruende der Naturwissenschaft* (1786),[1] will hardly contribute to an increase of respect for the philosopher of Koenigsberg.

Apart from these unhappy applications, the radical distinction between form and matter, or between the subjective and objective, cannot be accepted in the absolute sense in which our philosopher understands it. Solomon Maimon realized quite well in his analysis of the Kantian conception (1790) [2] that what we have here is only a relative separation of elements, distinguishable in a progress of successive approximations. Nor can we consider as valid the view of Kant according to which the *reality of the a priori principles* is proven by the *possibility of experience* which presupposes them. The dilemma which is assumed here between a standard-experience and the impossibility of any experience does not hold, since experience in reality offers us only degrees of knowledge, which are, to be sure, indefinitely perfectible.

[1] *Saemmtliche Werke* (Leipzig, 1838), vol. V, p. 303.

[2] Cf. *Versuch einer neuen Logik*, 1794, reprinted in Berlin in 1912. A new symbolic analysis is also developed there.

What the idealistic thesis thus amounts to in the main is a vigorous affirmation of certain mental requirements. In order that objects may be built up into a human science, they must lend themselves to the conditions of intelligibility and representability. And the value of this thesis—even when compared with positivism which confines knowledge to its objective content—can apparently be revealed only by an historical consideration of the development of scientific concepts. To this one may at most add an unlimited faith in the adaptability of experience to the regulative principles of reason. In illustration of this view we may refer to the distinction between the primary and secondary qualities of matter. If it no longer lays claims—thanks to the efforts of Berkeley—to metaphysical reality, it can nevertheless be maintained—in the light of reflective thought as well as in that of history—that it stands for a certain ideal of scientific thinking. Kant expresses this by attributing to the primary qualities alone the fact that they imply a priori conditions of objects.[1] For who would venture to maintain that the criterion of evidence has to conform to Cartesian concepts rather than to be worked out in harmony with a reflection supported by the widest experience?

The proper logical function of the understanding was studied by Kant not only in the second part of the *Critique of Pure Reason* but also in his lectures on *Logik*,[2] which were collected and published by Iaesche in 1800. It can be easily seen from this special work that the

[1] Cf. the concluding remarks of § 3 of the *Critique of Pure Reason*.
[2] *Werke*, III, p. 167.

Kantian conception of logic is as far from the clearness and preciseness of the Leibnizian as the view of thinkers not trained in mathematics is removed from that of mathematicians.

Leibniz was quite aware of the fact that "possibility" was the same thing as logical "reality"; he introduced the principle of sufficient reason in order to determine the world of existence (physical reality) over against the realm of entities or subsistence (that is, the logical world). Kant takes up these Leibnizian concepts, but he uses the principle of contradiction to determine "possibility" and that of sufficient reason to establish "logical reality." It is certainly hard to see what this latter concept can mean.

The confusion of logical and metaphysical criteria in the modality of judgments (possibility, necessity, contingency) has been already pointed out by the Kantian, W. Hamilton. But the absence of a clear recognition of what is logical or formal becomes especially obvious in the analysis which Kant gives us of such logical acts as *comparison, reflection,* and *abstraction.*[1] He does not ask himself under what conditions these acts are really independent of the particularities of things, or of the relationships which they may have with the preceding content of consciousness, or with the fundamental feeling dominating it. The Leibnizian lesson that the principles of contradiction and identity express the *invariance* of objects in *logical* thinking should have convinced him that a mental operation can be regarded as logical in one sense only, namely in so far as we associate or dissociate ideas which remain invariant during this process itself.[2]

[1] *Logik,* § 6.   [2] Cf. Maimon, *Versuch einer neuen Logik,* p. 15.

The weakness and especially the obscurity and the lack of precision in Kant's logic show themselves clearly in his theory of definition. Definition according to him (*Logik*, § 99) is "conceptus rei adequatus in minimis terminis determinatus." Nominal definitions are only explanations of names which contain the arbitrary meaning ascribed to certain names and which, consequently, do not indicate the essence of the object (§ 106). On the other hand, he distinguishes the *analytical* definitions of a concept that is given from the *synthetic* ones of a concept that is made. Strangely enough, our philosopher, whose title to glory rests upon his vindication of the activity of the mind, does not notice that *all* concepts are made. For concepts must be defined synthetically if we want to determine them precisely in logical thought, that is, they have to be constructed by the mind (in the same way as he finds in mathematics) after the image of that reality which is to be logically represented.

We shall, however, not dwell upon a fruitless examination. Kant's error in regard to the nature of definition, his confusion of formal and material criteria of knowledge, and finally all that is obscure and vague in his thinking will become especially evident to mathematicians looking at the rules for examining definitions given in § 108:

1. Whether it is true when regarded as a proposition;
2. Whether it is distinct when regarded as a concept;
3. Whether it is sufficiently distinct when regarded as a distinct concept.
4. Whether, finally, it is also determined, that is adequate, when regarded as a sufficiently distinct concept.

The obvious defects of the Kantian logic perhaps justify the suspicion of Venn [1] that the effect of our philosopher's logical speculations was as harmful as the influence of his philosophy was useful.

In view, however, of the special importance of Kant's historical position, it is necessary to resort to his philosophy in order to find out what the general conception of logic was going to become after it had ceased to be regarded as a real representation of an ontological classification. If we are not satisfied with the example of the Stoics and with that of modern symbolism which treat logic from a purely discursive and grammatical point of view, it seems that we have to reduce logic to psychology. It is worth while to quote Kant's remark [2] in this connection: "Logic is a rational science, not according to its matter, but only according to its form; it is an a priori science of the necessary laws of thought, but not in regard to particular objects but to all objects in general; it is thus a science of the correct use of the understanding and of reason in general, but not subjectively regarded, that is, not from the point of view of empirical (psychological) principles which establish how the understanding actually thinks, but objectively regarded, that is, from the point of view of a priori principles which determine how it ought to think." And he goes on criticizing those logicians who look in logic for psychological principles; this is for him as absurd as the attempt to derive ethics from life. It is indeed not a question here of finding out how the understanding actually thinks, confronted as it is by obstacles and all kinds of condi-

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[1] *Symbolic Logic*, 2nd ed., Introduction, p. xxxv.
[2] The end of § 1 of the "Einleitung" to the *Logik* (Werke, III, p. 175).

tions, and consequently of establishing contingent laws. What is of importance here is the discovery of the necessary rules; these which govern the understanding thought finds in itself without psychology.

This categorical admonition to distinguish between the logical and psychological point of view (in harmony with the whole method of the *Critique*) can be easily understood when one thinks of the empirical fact of psychological genesis. It proves then to be an invitation to keep in mind the ideal of logic as reflected in science, an ideal which has the meaning of a norm and which we find in our consciousness. Nevertheless, logic has also as its object the study of mental operations and of the laws which govern and must govern them when thought assumes the forms of the exact reasoning displayed, for instance, in mathematics. And this study of the activity of thought satisfying certain ideal conditions is on the whole a *rational psychology,* which is opposed to *empirical* psychology in the same way as the theory of motion of frictionless bodies is distinguished from the physics of motion.

This rational psychology certainly assumes significance through the social relations of human beings. It is precisely because science made expresses the collective work of society that the results of the critical regressive method acquire a social and human value as a "possibility of agreement" among different minds separated in time and space. In this way we may speak abstractly of an impersonal reason over and above the individual minds. But we have no right to go beyond this interpretation so suggestive of Comte, unless we are willing to fall back upon the assumption that logical relations are

real in the mind of God. Such a view is clearly expressed in the *Nouveaux Essays* of Leibniz (XXX, 4), which appeared in 1765 and which accordingly must have influenced the thought of Kant. It is permissible to suppose that it left traces in the mind of our philosopher. But it is exactly from this point that the further development of idealism started. In its romantic intoxication this school exalted the universal Spirit as an obscure constructive principle of the world, which was conceived to be immanent in individual consciousness after the fashion of the Platonic idea "One in many."

This metaphysical interpretation forces those who adhere to the positive significance of the doctrine and who do not wish to go beyond the concept of *formal* logic professed by Kant to admit (with Fries) that the analysis of the activity of thought—even when based upon a criticism of science rather than upon an empirical psycho-genetic method—has always, ultimately, a psychological meaning. And the objective value of a priori principles deduced from the possibility of experience can be justified, as we have said, at each historical stage only in so far as experience proves to be adapted to the requirements of reason.

The clear realization that logical activity is a definite function of the understanding, independent of the objects of thought, involves a virtual reform of logic itself. This is to reach its full development in the criticism of mathematicians during the nineteenth century.

THE REFORM OF CONTEMPORARY LOGIC

20. GENERAL REASONS FOR THE REFORM OF LOGIC IN THE NINETEENTH CENTURY

The development of logic before the nineteenth century did not apparently change the traditional conception of the structure of the demonstrative sciences; it only made their meaning clearer, as we see especially in the case of Pascal. The metaphysics underlying the Aristotelian logic, nevertheless, gives way to new modes of thinking. The ideal of science as expressed by Leibniz is essentially a revival of Platonism. On the other hand, the ontological presuppositions of ancient rationalism are made to retreat before the psychological criticism of knowledge. Logic thus becomes conscious of its formal character, being reduced to a doctrine of mental processes. It is in connection with this that we can understand the significance given to *nominal* definition, which is henceforth recognized by the best thinkers as the only definition proper.

But the defeat of mathematical rationalism, which shone so brilliantly in the seventeenth century, seems to entail a decay of logical thought. What is going to happen in the nineteenth century? Is there no reason to fear that a positive science independent of philosophy will kill the logical interest of the Renaissance bent above all on the *ars investigandi?* On the contrary, the reform of logic, which had been prepared for a long time, now

attains maturity under the influence of various factors connected with the development of mathematics.

But the historian who wishes to go deeper into the subject is confronted here with a particular difficulty due to the absence of a guiding criterion determining this development. In contradistinction to the unity of progress of the preceding centuries, which were always dominated by a central interest, there emerges now a plurality of problems and tendencies like the fragments of a demolished building or at least like the disjointed branches of a tree widening at the top in order to expose all its leaves to the sun.

It is easy to show the reason for this free development. Mathematics affirms its right to exist as a pure discipline, independently of its applications to the natural sciences. This claim finds its justification now in the maturity of certain theories and in the esthetic interest of the problems connected with them, now in the necessity of solving in a critical manner the doubts inherent in the use of certain concepts and of thus restoring to mathematical knowledge that firm certainty which is its pride and glorious tradition. But the philosopher will also discover, apart from this, the crisis following in the wake of the defeat of rationalism. The mathematician of the nineteenth century can no longer be helped by the faith in his ability to derive the solution of the problems of nature from the depths of his own thought. His intuition, moreover, is no longer subject to an external authority; his activity can and must therefore develop freely, shaping the framework within which the physicist will be able to arrange certain orders of phenomena in a formal way. But this does not mean that mathematical production

goes on unbridled and unchecked and that all the water of the great river is scattered and lost in a thousand rivulets. The development of thought obeys certain inner controlling forces, and in the different currents there can still be seen a reflection of traditional problems. The various specialized tendencies are in this way reunited in firm knots which give birth to higher doctrines. In short, that order which mind is not able to derive from external nature it finds in itself, in the full freedom of its activities. It is not, however, an order that is given; it is one that is progressively constructed.

The historian of logic has to take into account and to compare several movements of thought, which are of different origins, and which nevertheless react upon one another, converging, as we shall see, towards the same reformative view.

1) We shall first point out the birth of projective geometry in the school of Monge and Poncelet. The epoch-making work in this respect is the *Traité des propriétés projectives des figures* by Poncelet (1822). The systematic elaboration of projective properties is, however, a result of a long development which goes back to Desargues and Pascal and which is continued by the school of Newton and by several geometricians of a secondary order.[1] The formation of the concepts of projective geometry is accompanied in the school of Monge by a wide philosophical criticism extending, for instance, to the principle of continuity (Poncelet). It is this criticism that serves as a starting point especially for the logical work developed by Gergonne in his *Annales de Mathématiques*

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[1] Cf. Chasles, *Aperçu historique sur l'origine et le développement des méthodes en Géométrie* (Brussels, 1837).

between 1816 and 1819 and later in his philosophical considerations on the *principle of duality*, which was formulated by him in 1826. Considerable light was thrown upon this principle, which was gradually to assume a higher significance, by the more general considerations of transformations and correspondences (Möbius, 1827) and then by the use of coördinates introduced for representing any geometrical entities (Plücker, 1830).

2) The origins of non-Euclidean geometry [1] seem to be quite removed from projective geometry, although we nowadays consider with Klein that the former is, like the Euclidean, one of the metric systems which can be subordinated to the same projective system in relation to an absolute quadric. The classical researches on the Euclidean postulate of parallels lead to the establishment of the possibility of a geometrical system in which this principle is rejected. Gauss, Lobatchevsky (1829-40), and Bolyai (1832) join here in showing that this postulate is *indemonstrable*.

In the first place, the critical researches on the foundations of the theory of parallels led geometricians to a more refined conception of the value of postulates and induced them also to recognize and formulate hypotheses which are hidden implicitly in the *evidence* of principles. This is already seen in the works of the forerunners, Saccheri and Lambert. Criticism proceeds from this point to generalize these principles, assuming a tendency which becomes especially prominent in the theory of multi-

[1] Cf. R. Bonola, *La Geometria non-euclidea. Esposizione storico-critica del suo sviluppo* (Bologna, 1906). See also art. 8 of the same author in *Questioni riguardanti le matematiche elementari*, collected by F. Enriques (2nd ed., vol. I, 1912).

dimensional spaces (H. Grassmann, 1844, and B. Rie-
mann, 1854) and which expresses itself in more than one
sense in the works of Riemann and Helmholtz (1868).

In the second place, the recognition of a geometrical
possibility that does not agree with our intuition of space
gave the finishing stroke to the metaphysical rationalism
of the eighteenth century. Reality, even that reality
which corresponds to the Eleatic conception of rational
being, cannot be determined *a priori*. The fact is that the
choice among possible geometries depends upon an *ex-*
perimental verification, no matter whether we resort to
the exact measurement of geodesic triangles (in which
Gauss was interested) or to astronomic observations (as
Schweikart proposed already in 1817). And the best
mathematicians who accepted the thesis stated in these
terms had to admit that it also affected the Kantian doc-
trine of space: the famous bridge which was to lead us
to the land of absolute idealism.

The influence of non-Euclidean geometry on the logic
of the nineteenth century does not stop here. We have
to take into account, in the third place, that phase of
the development of the doctrine (attaching itself to
multi-dimensional constructions) in which we are in-
terested in its *possible concrete interpretations*. This idea
was suggested by Riemann and was worked out by Bel-
trami in his classical "Saggio d'interpretatione della geo-
metria non-euclidea" (1865). It is by starting from this
important work and from that of Hesse on an "Uebertra-
gungsprinzip" that the concept of *abstract geometry* was
developed in all its fullness. Here one can find a natural
extension of the principle of duality derived from pro-
jective geometry, and also an immediate foreshadowing

of the new conception of the *hypothetico-deductive sys-tem* of contemporary logic.

3) The development of logic in England arrived at the same abstract conception of formal science by an independent and quite different road. The vast move-ment of mathematical logic that centered in that country in the fundamental problem of expressing thought by means of symbols [1] seems to have been influenced by the adoption of the continental symbols of the differ-ential and integral calculus; for the old Newtonian nota-tions were still used at the beginning of the eighteenth century. The progress achieved by celestial mechanics, thanks to Laplace, and in general the work of the great French analysts were chiefly responsible for this reform, the result of a real crusade undertaken at Cambridge, notably by Whewell. But other doctrines that originated on the continent also penetrated into England, where they called for a similar elaboration. We may mention in particular the theory of imaginaries by which Pea-cock (1833) [2] arrived at the analysis of the formal prop-erties of operations, and then the calculus of Probability of Condorcet as well as the first applications of the mathe-matical method to statistics and economics which at-tracted the attention of G. Boole and S. Jevons. These names of the founders of English symbolic logic figure beside those of A. De Morgan and W. Rowan Hamilton, the author of the theory of Quaternions. The latter must not be confused with the Kantian philosopher W.

[1] Entirely independent of the unknown researches of Leibniz, Segner, Lambert, etc., of which we have spoken above in § 19.

[2] "Report on the Recent Progress and Present State of Certain Branches of Analysis." Rep. on the Third Meeting of the British Assoc. for the Advancement of Science held at Cambridge, 1833. London, 1834 (pp. 185-352).

Hamilton, whose doctrine of the quantification of the predicate is of no great importance for symbolism.[1]

4) The true value of the abstract concept of formal theories could be appreciated only by connecting it with the tendency of the positivistic philosophy which aims to free physical doctrines from the underlying metaphysical hypotheses and to see in them merely *models* (in general, mechanical models) of reality.

When this positivistic tendency penetrates into the logical criticism of the principles of mathematics, it becomes intertwined in a peculiar and interesting way with the influence of non-Euclidean geometry. The idea, for instance, of the reciprocal translation of diverse theories leads in general to an examination of the intuitive data which tend to be extended to different forms of intuition. This gives rise to a special *interest* in a more thorough analysis of certain *evident principles,* whose very evidence would have stopped any scrutinizing criticism.

This is brought out clearly by the examination of the *axioms of equality* undertaken by Mach and Maxwell (before Helmholtz). Both of these thinkers pointed out that the axioms in question represent a translation—in a form evident in regard to certain orders of concepts— of physical *facts,* whose verification forms a necessary condition implying that reality admits truly of such a conceptual representation.

5) There were also other factors in the development of mathematics in the nineteenth century that concurred

[1] Venn considers it as a petty reform; he also contests its priority (which belongs to Plouquet) and blames its author for the inexact interpretation of the schemata used by him (cf. op. cit., p. 9). Cf. also the polemic with De Morgan which extends from 1846 to 1873 in the *Athenæum* and in the *Contemporary Review.*

with the non-Euclidean geometry and the positivistic interpretation of physics in bringing about the criticism of evident principles. There was in the first place the need of supplying analysis with a solid foundation. To this end it was necessary to remove the difficulties connected with the infinitesimal calculus and to solve the paradoxes offered by divergent series, the pseudo-demonstrations of maxima and minima, of derivatives, etc. From Abel to Cauchy and Weierstrass, in whom the *arithmetization* of mathematics culminates, all the great analysts had devoted themselves to this work, which finally resulted in the systematization even of the most complicated parts of this science, and in finding for it a most rigorous basis in the fundamental concept of *whole numbers*.

On the border line of this movement there flourished the more philosophical criticism of Bolzano, du Bois-Reymond, Cantor, etc., which has given us a new *analysis of the infinite*. Here the limits of evidence are surpassed and the significance of the *axioms of inequality* is made to appear in a new light.

6) But despite the great importance of these various movements, which all concur in the *reform* of contemporary *logic*, we believe that this reform asserts itself fully only in the *most recent criticism of the principles of geometry*, through which mathematicians become fully aware of the significance of the revolution prepared by the work of centuries.

We have dwelt upon certain points which we intend to develop in the following sections. We shall survey

rapidly the most characteristic aspects of the motives pointed out above. In this way we shall try to explain the true meaning of the reform springing from these motives as well as of the whole movement of thought which we have to regard as correlative to the development of mathematics, and especially to the criticism of the principles of science.

21. THE PRINCIPLE OF DUALITY AND THE LOGICAL WORK OF GERGONNE

We have already said that the birth of projective geometry was accompanied by the logical criticism developed by Gergonne (1816-1819) in the *Annales de Mathématiques,* to which he later added the formulation of the "principle of duality" (1826). The articles written by Gergonne in the period between 1816-19 deserve attention; we shall, therefore, enumerate them here.

The "Essai de dialectique rationelle" (vol. VII, p. 189) starts from the so-called *circles* of Euler [1] and tries to derive the classification of syllogisms from the rules of conversion of propositions. The essay, "De l'analyse et de la synthèse dans les sciences mathématiques" (ibid., p. 345), places the basis of theorems in axioms and that of problems in postulates, and explains the meaning of analysis and synthesis. On the whole, the lucid exposition does not seem to rise much above the level of traditional logic, but it is noteworthy that our author ex-

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[1] This way of representing subordination of concepts, expounded in the *Lettres à une Princesse d'Allemagne,* can be found already in Johachim Jungius, the teacher of Leibniz, as was shown by Itelson and by Vailati (*Scritti,* p. 621). But it goes really back to Ludovicus Vivès (De Censura Veri: *Opera,* p. 607), quoted by Venn, op. cit., p. 507.

presses twice his conviction that all definitions are nominal (cf. the notes on pages 346, 364). Finally the "Essai sur la théorie des définitions (vol. IX, p. 1) contains some highly original thoughts.

After remarking that definitions introduce only words in order to denote certain sets of ideas, the author clearly sets forth the rules governing them, and in this way he is led to make certain interesting observations. That he fully grasped the importance of nominal definitions can be seen, for instance, from rule III, where he says that names should be given only to those sets which are expected to recur often in discourse; definition is thus conceived as the seal of a constructive process of thought. But a more original observation is to be found in the rule which says that definitions must contain no other unknown terms besides the ones to be defined, for to define by means of unknown words is, according to the author, equivalent to an equation between two unknown expressions. In this way he is led to give a wider scope to definition. "If a proposition contains a single word whose meaning is unknown to us, the enunciation of the proposition is sufficient to reveal its meaning to us. If some one, for instance, who knows the words "triangle" and "quadrilateral," but who never heard the word "diagonal," is told that each of the two diagonals of a quadrilateral divides it into two triangles, he will understand at once what a diagonal is and he will understand it all the better as this is the only line by which a quadrilateral can be divided into triangles. Propositions of this kind, which give the meaning of one of the words contained in them in terms of others that are already known, can be called *implicit definitions*, in contradistinc-

tion to ordinary definitions, which can be called *explicit definitions*. We can also understand that . . . two propositions which contain two new words, combined with known terms, can often determine their meaning" (pp. 22-23).[1]

The theory of implicit definition of a system of concepts by means of a system of propositions has become essential for contemporary logic. But this theory would not have attained its present position without the light thrown on it by the principle of the substitution of concepts, which has its germ in the principle of duality of projective geometry. Omitting an older observation of Snellius in regard to the symmetry offered by the theorems of spherical geometry, we shall treat briefly this fertile principle, which Gergonne formulated in the "Considérations philosophiques sur les élements de la science de l'étendue" (*Annales*, vol. XVIII, p. 125, January, 1826), and of which, as he says, he had had a clear notion since 1819.

After distinguishing between geometry of position and metric geometry (graphical and metrical properties of figures), the author remarks that the theorems of geometry of position (which are not dual by themselves) occur always in pairs. Each theorem may be converted into its dual by replacing "points" by "straight lines" in the case of plane geometry, and "points" by "planes," in the case of geometry of space, and by leaving the "straight lines" unchanged. Such symmetry or duality constitutes for Gergonne an *a priori* principle, which makes possible a "géométrie en parties doubles" so that given a theorem of position its reciprocal can always be

[1] Cf. G. Vacca, *Rivista di Matematica*, 1899.

enunciated and regarded as true. Our author shows how such a geometry can actually be developed by starting from the simplest principles, how dual propositions can be demonstrated in a perfectly symmetrical manner by writing them in two opposite columns, as is still done to-day in treatises of projective geometry.

It would not be out of place to make a few historical and critical remarks at this point. Dual propositions must have attracted the attention of students of projective geometry long before the publication of Gergonne's essay. Suffice it to mention that Brianchon had already in 1806 deduced the theorem of a hexagon circumscribed about a conic from Pascal's theorem (dual) of an inscribed hexagon by means of a polar transformation of the figure. It was only in 1824 that Poncelet formulated the general method of reciprocal polars. There can be no doubt that this transformation virtually contains the duality of geometry of position, which also finds an *a posteriori* demonstration in the transformative instrument. This accounts sufficiently for the claim of priority which was published in the same *Annales* (vol. XVIII, p. 125). But this does not diminish, in our opinion, the value of Gergonne's conception which postulates a duality *a priori*. The merit of Gergonne (and especially his philosophical merit) is not seriously impaired even by the errors committed by him in the application of the principle. There was a time [1] when he believed that to an algebraic plane curve of the order n there corresponds by duality a curve of the same order. The author later corrected the error at the suggestion of Poncelet. (*Annales*, vol. XVIII, p. 152.) But Poncelet, who dis-

[1] *Annales*, vol. XVII, pp. 216-219.

covered the error, did not quite understand how duality could be applied to curves of an order higher than the second.

It is more important to understand in what sense and to what extent Gergonne justified his own principle. No modern critic could fail to notice here an obvious gap. Our author, it is true, shows us the parallel development of the first theorems of geometry of position out of simple principles. His analysis of these, however, does not go so far as to bring out the fact that they form a system of postulates sufficient to serve as a basis for a geometry of position. He has therefore no right to conclude *a priori* that *all* the theorems of geometry of position must possess that logical symmetry which the principles satisfy. At that time it was all the more difficult to fill this gap as the structure of projective geometry seems to have been connected with Poncelet's method of projection, which reduces figures to special metrical cases. Gergonne's conception in reality presupposes the later development of projective geometry as it was worked out by Staudt (1847) independently of the employment of metric notions. It is only through this development that the principle of duality becomes truly established *a priori* and it is in this way that the idea of the French philosopher is fully realized.

Gergonne's declaration and Poncelet's polemic had meanwhile their effect on contemporary geometricians who were above all interested in establishing the fact. Möbius (1827) and Plücker (1830) solved the problem. Möbius,[1] who has given mathematical science the most general concept of correspondences and transforma-

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[1] *Barycentrische Calcul,* p. 436.

tions, establishes the symmetrical character of the reciprocal relation obtaining between two dual elements (point and straight line in a plane, point and plane in space). Through this it is possible to prove that any reciprocity (and not only the polarity considered by Poncelet) transforms one figure into another possessing dual properties. Plücker [1] bases the principle of duality upon the consideration of coördinates and planes, a method enabling us to treat correlative relations in an identical analytical manner. This view contains in germ the widest extension of the principle of duality, conceived as "a principle of infinitely possible interpretations of abstract geometry," of which we shall speak in the next section.

22. ABSTRACT GEOMETRY

We have shown how the development of non-Euclidean geometry and of multi-dimensional doctrines gave rise to an *abstract geometry susceptible of several interpretations* and containing—in a higher sense—the principle of duality of projective geometry.

Virtually this concept is already wholly contained in Plücker's method, according to which we represent by means of coördinates not only points, straight lines, and planes, but also any figure that is capable of varying continuously, as a function of certain parameters. The analytical properties of sets of trios (x, y, z) are reflected either in the figures of the space where these trios are regarded as coördinates of their generating elements,

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[1] *Analytisch-geometrische Entwickelungen*, part II. Cf. *Abhandlungen*, vol. I, p. 619.

"the points," or in the systems of circles of a plane,
in case these numbers are assumed as coefficients of the
equation of a circle, that is, as coördinates of circles.
But the analyst who reasons in this way aims to reduce
systematically the geometrical difficulties inherent in the
study of different kinds of figures to the universal lan-
guage of calculus. The direct comparison of two orders
of geometrical properties, or of two geometries which
are unified in the analytical representation, leads still
further; it invites us *to translate different forms of intui-
tion into one another*.

We can now answer in the following way the question
whether non-Euclidean or multi-dimensional geometry
(independently of the metaphysical possibility that can
be discovered here) is a pure scheme of algebraic for-
mulas: "Infinite orders of geometrical properties be-
longing to entities of our Euclidean space can be regarded
as interpretations of a non-Euclidean geometry or of a
geometry of more than three dimensions." Thus in the
example of Beltrami the non-Euclidean geometry of a
plane surface is reflected in the geometry of curvilinear
figures drawn upon a surface of negative curvature,
where the geometrical line takes the place of a straight
line. And according to Klein, the system of straight lines
of our ordinary space represents an image of a four-di-
mensional manifold of the second order, immersed in a
five-dimensional linear space.

It was exactly through Klein and Lie that the concept
of abstract geometry received its great development, be-
coming later (after Serge) an ordinary working instru-
ment in the hands of contemporary Italian geometricians.
Nothing is indeed more fruitful than the increase of our

intuitive powers made possible by this principle. It seems as if to the mortal eyes with which we examine a figure under a certain aspect there were added a thousand spiritual eyes enabling us to contemplate so many different transformations.

But in order to use such a principle fruitfully it is necessary for our logical faculties to be exercised in a sure manner. We pass from a system A to a system B, the two systems appearing as possible interpretations of the same abstract theory. This means that certain relations *a* of A can be translated into certain relations *b* of B, and that, therefore, all the logical consequences of the *a*'s entail similar consequences of the *b*'s, so that we acquire *a priori* new knowledge concerning B. In order to discover the consequences of the *a's,* we *consider* the objects which are given to us with A and we have to ask whether the properties discovered in this way do not require a new intuition the equivalent of which might be absent from B, or whether they cannot be derived by logical deduction from the relations *a* without the possible addition of anything evident. The instrument of abstract geometry is thus based upon a constant repetition of the *logical analysis of the principles of deductive theories* in accordance with the various systems of concepts and the various forms of intuition. And it is through this exercise that geometricians have nowadays developed *the meaning of what is logical* to a degree that could not be attained by others.

23. The Concept of Formal Science and of Its Different Interpretations in the Works of the English Mathematical Logicians

There is a close similarity between the idea of abstract geometry and the free concept of the formal and speculative sciences entertained by the English mathematical logicians in opposition to that of the physical sciences.

"In the speculative sciences," Peacock says in the above mentioned report of 1833,[1] "we merely regard the results of the science itself and the logical accuracy of the reasoning by which they are deduced from assumed first principles; and all our conclusions possess a necessary existence, without seeking either for their strict or for their approximate interpretation.

"In the physical sciences we found our reasonings equally upon assumed first principles and we equally seek for logical accuracy in our deductions from them; but both in the principles themselves and in the conclusions from them we look to the external world as furnishing by interpretation corresponding principles and corresponding conclusions . . .

"The first principles which form the foundation of our mathematical reasonings in the physical sciences, being neither arbitrary assumptions nor necessary truths, but really forming part of the series of propositions of which those sciences are composed, can never cease to be more or less the subject of examination and inquiry at any point of our researches . . . But in the abstract sciences of geometry and algebra those principles which are the foundations of these sciences are also the proper

[1] Loc. cit., p. 187.

limits of our inquiries; for if they are in any way connected with the physical sciences, the connection is arbitrary and in no respect affects the truth of our conclusions, which respects the evidence of their connection with the first principles only and does not require, though it may allow, the aid of physical interpretations."

It is easy to see how this formal concept of mathematical science leads to the view that one and the same abstract theory can receive different interpretations.

This view (whose meaning is no less wide than that of abstract geometry) seems to have been expressed for the first time by Gregory [1] (1840) in regard to the algebra of logic. Boole (1847)[2] states it in the following manner: "Those who are acquainted with the present state of the theory of symbolic algebra are aware that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed but solely upon the laws of their combination. Every system of interpretation which does not affect the truth of the relations supposed is equally admissible, and it is thus that the same process may under one scheme of interpretation represent the solution of a question or the properties of number, under another that of a geometrical problem, and under a third that of optics."

And after insisting upon the importance which such a principle has for analysis, Boole regrets that its application has been restricted to the case in which the elements to be determined are conceived as measures (as the example of Plücker's principle shows). A beau-

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[1] *On the Real Nature of Symbolic Algebra*. Trans. of the Royal Society of Edinburgh, vol. XIV.

[2] G. Boole, *The Mathematical Analysis of Logic* (Cambridge, 1847). Introduction, p. 3.

tiful application which the author himself and other logicians [1] have made of symbolic analysis consists indeed in the establishment of a *logical calculus* of events parallel to the numerical calculus of *probability*. If x, y . . . are the events studied and p_x, p_y are the numbers which denote their probability, the operations on p_x, p_y . . . can be interpreted as logical operations on x, y . . . To the probability of the conjoined events:

$$p_x \text{ and } p_y = p_x \, p_y$$

there will thus correspond the *logical product*

$$x \text{ and } y \; (= x \times y),$$

while to the probability of the occurrence of one of the two events

$$p_x \text{ or } p_y = p_x \, p_y,$$

there will correspond the *logical sum*

$$x \text{ or } y \; (= x + y)[2]$$

24. Positivism and the Criticism of the Axioms of Equality

The free conception of formal science which we have found in the English mathematical logicians corresponds exactly to the use made by the physicists (Maxwell, Lord Kelvin . . .) of *mechanical models*. As we have pointed out, we see here the general influence of positivism which reveals itself indirectly in the criticism of logical concepts in the following manner.

[1] A. De Morgan (1847), Boole (1854), Peirce (1867). Cf. P. Medo-laghi, "La logica matematica e il calcolo delle probabilità." *Bolletino dell' Ass. Attuari,* Nov., 1907.

[2] See the analysis of symbolic logic in § 28.

There are physical theories in which visible phenomena are made to depend on quantities supposed to represent something hidden. To those who accept their metaphysical basis the axioms concerning such quantities are immediately evident. But to those who disregard the hypothetical basis of the theory and look only for its positive content these axioms appear as general facts, whose subsistence is a *condition* for the applicability of certain representative concepts to a certain order of phenomena. This observation throws light upon the nature of the *evidence* of logical principles and especially upon the concept of equality.

It seems that Ernst Mach was the first to embark upon the new criticism which we are going to consider here. What is the meaning of the statement that "two masses that are equal to a third mass are equal to each other?"

Such a question would not have been put by a Galilei or a Newton, who accepted the Democritean hypothesis of a homogeneous substance of which the atoms are made (they differ in shape and volume only). The term "mass" denoted for them nothing but "quantity of matter," or the total volume of the atoms forming a body. The principle, when applied to volumes, would thus reduce itself to an evident geometrical axiom. But Mach rejects the metaphysical hypothesis of the homogeneity of matter and he wants to define mass or the "relations of masses" by means of actual experiments in which the masses to be measured are compared in pairs. How is it possible under such conditions to maintain *a priori* that the relations measured are independent of the choice of the unit of measurement or that "masses equal to a third are equal to each other?"

The positivistic philosopher realizes quite well that what we have here is not logical (or geometrical) necessity but only a physical fact whose verification is a condition making possible the very concept of mass, which is to be defined. This observation of Mach goes back to 1868.[1] Clerk Maxwell, on the other hand, independently developed the same criticism in regard to "temperature" in 1871.[2] Here too, if the equality of temperature is defined in a positive way, by means of an experiment of thermic equilibrium, it cannot be maintained as *a priori* evident that "temperatures equal to a third are equal to each other." To speak in Kantian terms, this judgment which seems to be "analytic" turns out in reality to be "synthetic."

The views of Mach and Maxwell paved the way for a more thorough criticism of physical as well as of geometrical equality, which was developed—in a highly philosophical sense—by Hermann Helmholtz.[3]

Helmholtz grants Kant that space can be an a priori form, but he denies that the axioms are also a priori. He especially points out the physical and positive meaning of geometrical equality. Physically equivalent magnitudes, he says in reply to Land (1878), are those in which there can occur the same processes under the same conditions and in the same periods of time. As a means

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[1] Carl's Repertorium, vol. 4, 1868. Cf. *Erhaltung der Arbeit,* 1872, *Die Mechanik in ihrer Entwickelung.* Ch. II. V, 4th ed. Trans. into English by T. J. MacCormack as *The Science of Mechanics* (Chicago, 1907).

[2] *Theory of Heat,* 9th ed. London, 1888. (1st ed., 1871.) Cf. Mach, *Die Prinzipien der Waermelehre* (Leipzig, 1896), p. 39.

[3] Cf. his criticism of the principles of geometry in *Wissenschaftliche Abhandlungen.* Vol. II, pp. 610, 660 (1866, 68, 78), *Zaehlen und Messen, erkenntnistheoretisch betrachtet* (1887), ibid., vol. III, p. 356.

for determining equivalence he uses the motion of rigid bodies. For the equivalence of two physical magnitudes assumes an objective significance owing to the fact that an equivalence established by one method can also be established by any other method. In this way we are enabled to regard *geometry as a physical science*.[1]

Our author comes back to the question of physical equality in a later work, where he arrives at the following conclusion. When we wish to isolate an attribute of bodies in our mind, *by abstraction*, and to compare the bodies under a certain aspect in order to establish thus a relation of "equality," the physical relation in question must satisfy the condition:

from $\qquad\qquad a = c$ and $b = c$,

there follows $\qquad\qquad a = b$.[2]

This most general postulate, he remarks, must be presupposed wherever we wish to define equality with Grassmann,[3] by calling "equal" that of which the same thing can be predicated, or, in more general terms, what can be substituted in every judgment (of a given kind).

The preceding analysis attaches itself to the general criticism of the combinatory properties of *relations* made by De Morgan.[4] A relation is called by this thinker *transitive* when, under the supposition that it subsists

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[1] Loc. cit., vol. II, p. 648.

[2] Loc. cit., vol. III, p. 375 ff.

[3] *Die lineale Ausdehnungslehre* . . . , 1844; 2nd ed., 1878. Cf. also R. Grassmann, *Die Formenlehre der Mathematik*, 1872.

[4] *On the Symbols of Logic* (Transactions of the Cambridge Philosophical Society, 1856, p. 104. Cf. X, p. 345).

between a and b and between b and c, it also subsists between a and c. Transitive relations are more general than those that can be regarded as "equalities." For instance, the relation "ancestor of" or "descendant of," and also the relations of magnitude "larger than" are transitive, although they are not (like equalities) *convertible* and *symmetrical*. Indeed, if Peter is the ancestor of Paul, Paul is a descendant but not an ancestor of Peter; and if a is larger than b, b is not larger but smaller than a.

Symmetrical and transitive properties (from which we deduce the property designated by Vailati [1] as *reflexive*, implying $a = a$) characterize the *relations of equality*. For it can be admitted that entities connected by such a relation possess a certain property in common, giving rise to a concept which is a logical function of the entities in question and which is in this way *defined by abstraction*.

The employment of definitions by abstraction goes back to the fifth book of Euclid where we find an exposition of the theory of proportions of Eudoxus of Knidos. The relation ($\lambda\acute{o}\gamma o s$) between two magnitudes is indeed defined here by means of "proportion" or "equality of relations." The relation $a:b$ is said to be equal to the relation $c:d$, if, as a result of the multiplication of the given magnitudes a, b, c, d by the numbers m, n, whenever we have

$$ma > nb, \ ma = nb, \ ma < nb$$

we also have

$$mc > nd, \ mc = nd, \ mc < nd.$$

Similarly, by starting from the reflexive, symmetrical, and transitive relation of the parallelism of two straight

[1] *Rivista di Matematica* (1891), Cf. *Scritti,* p. 8.

lines we can define "the direction" which is common to a
system of parallel straight lines, etc.[1]

In the above cited cases ordinary language expresses
the relation obtaining between the entities a and b by
saying that the abstract concept which is a logical func-
tion of a is equal to the analogous logical function of b.
We say indeed:

> the straight lines a and b are parallel, that is:
>   direction of $a =$ direction of b.

But in other cases language itself consents to regard
as equal simply a and b, without mentioning the abstract
concept, although two modes of expression are used indif-
ferently; for instance:

> segment $a =$ segment b, or
> length of segment $a =$ length of segment b,

polygon $a =$ polygon b (in shape or magnitude) or:

> shape of $a =$ shape of b,
>   area of $a =$ area of b.

In accordance with this second manner of expression
one can give the most general definition of the *equality
of two entities* a and b relative to a group of properties
or to the class of which they are regarded as members.
The objects a, b, c, etc., of a class (a and b and c, etc.)
are said to be equal with reference to the class in which
they are contained.

In other words, we assume here that these objects can
be substituted for one another with regard to the abstract
concept of the class (a or b or c . . .), this concept

[1] Cf. Vailati, *Scritti*, p. 219. Burali-Forti, *Logica Matematica*, 1st ed.
(1894), p. 140.

being precisely defined in this way by means of the association of the objects a, b, c, etc., that is, by means of the equality established among them. This way of conceiving equality and definition by abstraction, which was the result of the criticism of the above mentioned physicists, can be found essentially in the analysis of the concept of cardinal numbers started by Cantor and Frege (1884), then further developed by Russell (1903) and also by Enriques (1912)[1].

25. The Analysis of the Infinite and the Axioms of Inequality

Owing to a rigorous criticism of infinitesimal analysis it became at last possible to banish from this science, and hence apparently from mathematics, the concept of the infinite and of the infinitesimal (actual), by reducing their theorems to simple considerations of indefinitely increasing and decreasing variables. At the same time a more philosophical investigation of the principles of arithmetic succeeded in solving the riddle of the infinite, and in removing the difficulties and paradoxes which had

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[1] Symbolic logicians often prefer to retain for equality the absolute meaning of identity; they thus write $(a) = (b) = (c)$ . . . instead of $a = b = c$, etc. But the function has to be regarded as a primitive logical concept defined by a mental operation—which is the inverse of the one combining a, b, c, . . . in one and the same class; this permits us to pass from a class to any of its members. Otherwise that function remains undetermined. It is this that is responsible for the difficulties which some symbolic logicians have found in definition by abstraction. Cf. Peano, *Formulaire de Mathématiques* (1901), p. 8; Russell, *The Principles of Mathematics* (1903), p. 219; Burali-Forti (*Rendic. Acc. Lincei* (1912) and *Logica Matematica*, 2nd ed. (1919). Enriques Burali-Forti, "Polemica logico-matematica" in *Periodico di matematiche*, no. 4, 5 (1921).

troubled thought since antiquity. We shall describe here briefly the development of the ideas which brought about this fundamental result.

Already Galilei had been struck by the paradoxical observation that the totality of natural numbers can be regarded as equal in number to one of its parts. Every natural number, for instance, can be made indeed to correspond to an even number, which is its double, or to a square, etc. From this paradox Galilei (*Opere*, VIII, p. 78) simply deduces that the attributes of "larger," "equal," and "smaller" cannot be applied to infinites. Cauchy goes still further in his lectures edited by Moigno in 1868. He admits that the paradox proves the impossibility of regarding an infinity of objects as existing together and as forming a unique whole. For it seems to him that the axiom "the whole cannot be equal to one of its parts (proper)" cannot be denied without contradiction. An explanation in an entirely opposite sense was attempted by Bernard Bolzano. The figure of this thinker and his philosophical views deserve here special mention.

Bolzano,[1] who attracted public attention in his day by his attempts to reform Catholicism, utilized the ideas of scholastic realism in a form greatly suggestive of the one found in the *Monadology* of Leibniz. In his *Wissenschaftslehre* [2] (vol. I, p. 77) he defines "propositions in

[1] 1781-1848. An Austrian priest and philosopher; he taught at the university of Prague that Catholic theology is in full harmony with reason, until his doctrines were condemned by the Church in 1820.

[2] *Wissenschaftslehre, oder Versuch einer neuen Darstellung der Logik,* in 4 volumes, Sulzbach, 1837, 2nd ed. (Leipzig, 1914). For the influence of Bolzano's logicism upon more recent thinkers see Th. Ziehen, *Lehrbuch der Logik* (Bonn, 1920), p. 173 ff.

themselves" as those propositions which are neither thought nor eventually expressed in discourse, to which he, however, applies the predicates of truth and falsity. For instance, the proposition "an equilateral triangle is also equiangular" would be regarded by him as "true in itself," even if no one ever thought of it or understood it. Propositions in themselves denote for Bolzano an "Aussage," which forms an object of thought and discourse but which "ist ueberhaupt nichts Existierendes." This statement is of course to be interpreted in the light of the metaphysical distinction between subsistents and existents (sense). There can be no doubt that Bolzano superimposes a world of Platonic ideas upon the world of phenomenal reality.

The *Paradoxien des Unendlichen*, which Bolzano started in 1847 and which were published only after his death,[1] contain the results of long meditations over the problem of the infinite. One can discover there the influence of the logical ideas of which we spoke above. Our author accepts the following idea of Leibniz: [2] "I have such a predilection for the actual infinite that instead of admitting that nature abhors it, as is commonly believed, I think that nature prefers it everywhere, in order to show better the perfections of its Author." It seems that this view necessarily imposes itself upon those who try to define concepts realistically, in accordance with an order of decreasing generality. For the concept of class, or aggregate or finite number presents itself then as a specification of a "summum genus," which also comprises the aggregate or the infinite number.

The philosophical presupposition of Bolzano thus leads

[1] Leipzig, 1851.   [2] Ed. Dutens, vol. II, p. 243.

him to remove, as far as it is possible, the contradictions attached to the infinite, although he has to admit that he has not succeeded in "showing that the appearance of this contradiction is only an appearance" (op. cit. § 1). The somewhat obscure conclusion reached by him seems to be the following one. There exist classes of entities in the case of which equality and sum can be defined in such a way that the sum $a + b$ is never equal to a, and these classes are finite (§ 6).

On the other hand, there exist (contrary to the view of Cauchy § 12) also infinite series and classes. Any doubt concerning the objectivity "Gegenstaendlichkeit" of the infinite can be removed through the observation that "the aggregate of propositions or of truths in themselves" is equally infinite. For if A is a proposition, the statement "A is true" also forms a new proposition, and similarly the statement "it is true that A is true," etc. (§ 13). The author combats then the view of those who believe that an aggregate cannot be regarded as given unless there is some one who thinks of it ("would any one deny that there exist at the poles of our globe bodies: water, air, stones, even if no human beings or other thinking creatures were to be found there?"). He also finds it strange that "the possibility of being thought" is assumed by some thinkers as a basis for the possible existence of things. For him, on the contrary, possibility is prior to thought: what is possible must be able to become an object of thought (§ 14). It is in the light of these criteria that Bolzano examines the question of the infinite. After showing in various ways the marvelous property possessed by an infinite aggregate which enables it to be put into a one-to-one correspondence to one

of its parts (§ 20), he explains that the paradox is due to the difference obtaining between the concept of the finite and that of the infinite (§ 22).

Bolzano's attempt to establish a calculus of infinites was less fortunate. He committed here the error of regarding as infinites of different powers what Cantor later discovered to be of the same power. His work, none the less, paved the way for Cantor.

George Cantor follows in the foot steps of Bolzano. He starts exactly from the idea that the paradoxical properties of the infinite are not due to an intrinsic contradiction but solely to the difference between the infinite and the finite. After examining impartially the properties of correspondence which can be established between infinite aggregates, he arrives at other marvelous and also consistent properties and succeeds in showing that every appearance of contradiction, to use Bolzano's terms, is henceforth only an appearance. The result obtained by him between 1878 and 1883[1] marks an epoch in the history of human thought; like the analysis of the infinite in antiquity, it is an event of intimate concern to logic.

This event overshadows the fact that the progress was attained here through a realistic philosophical view just as the critical systematization of the infinitesimal calculus was, on the other hand, achieved through the nominalistic mentality of Cauchy. Realism and nominalism or idealism and empiricism—to accept the almost equivalent terminology of Du Bois-Reymond—are two mental attitudes which appear historically to be connected with

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[1] Cf. especially *Crelles Journal*, vol. 84, *Acta Mathematica*, vol. 2, and *Mathematische Annalen*, vol. 46, 49 (1895-1897).

the origins of certain mathematical theories. They can, however, be considered as different interpretations applied to doctrines already formed, as can be seen in Paul Du Bois-Reymond's dialogue [1] between an idealist and an empiricist. Yet to believe that Cantor's theory of aggregates is necessarily connected with the realistic suppositions of its author would imply on our part a too narrow and one-sided view of it. On the contrary, every one can nowadays regard it as an extension of the possibilities of thought.

We shall try to explain this theory in its main features in accordance with the special exposition we have given of it in the article on "Real Numbers" published in the volume of the *Questioni riguardanti le Matematiche elementari*.[2]

It would be helpful to start here from the concept of number. Natural numbers 1, 2, 3 . . . have a double significance. They can be regarded as *cardinal* and as *ordinal* numbers according as they denote "the number of the objects of a class" or "the number of the order of an object in a series." But the two kinds of numbers correspond to each other in so far as we can arrange the classes in an order of succession, by taking first the class that contains one member, then the one containing two, and so on.

Let us turn our attention to cardinal numbers. The axioms concerning these numbers present themselves under a double aspect, as laws of association of thought or as expressions of the elementary properties of classes of objects. It is this circumstance that gives rise to the question whether they are a priori or a posteriori. This

[1] *Die allgemeine Funktionenlehre* (Tuebingen, 1882).
[2] Bologna, 1912.

can be solved in the one or in the other way, according as we stress the subjective or the objective aspect of these principles. Ultimately these aspects coincide, a fact which makes possible the application of logic. For instance, the axioms of the equality of numbers, objectively considered, represent certain elementary operations performed upon groups of objects, where a one-to-one correspondence is established between the members of two groups associated in pairs. Similarly, the axiom of inequality "the whole is larger than the part" means that it is impossible to establish a one-to-one correspondence between a group or class and one of its parts (proper). But this property presupposes that the class in question is *finite*. It does not hold of *infinite* classes; thus the class of all natural numbers (1, 2, 3 . . .) can be put into one-to-one correspondence to the class of even numbers (2, 4, 6 . . .) or to that of squares (1, 4, 9 . . .), although the latter are contained in the former. Has this circumstance, which finds its expression in the classical paradoxes of infinity, to be interpreted so as to imply that the concept of the infinite is intrinsically contradictory? Such a consequence is unavoidable if the axiom "the whole is greater than the part" is to be regarded as a purely analytic logical judgment. But it does not follow in case the axiom is taken in a synthetic sense, as expressing a property of the given classes. This is exactly the interpretation given by Bolzano and accepted by Cantor, an interpretation enabling the latter to establish the fact that the logical existence of *infinite cardinal numbers* can be conceived without contradiction. These present themselves as "powers of classes or of mathematically given aggregates," since the possibility of a comparison by means of

a one-to-one correspondence does not stop with finitude, although the special properties of the "finite" naturally no longer hold. Indeed, if we can still regard as "equivalent" two classes (infinite) which can be put into a one-to-one correspondence, and if we thus can call the respective numbers (powers) "equal," we have then to say that an infinite class is equivalent to one of its parts, and that equal numbers correspond to the whole and to the part.

This does not, however, preclude the possibility of essentially different infinites one of which can be said—in a proper sense—to be greater than the other. The power of the continuum (the number of points in a segment) is, for instance, greater than that of a denumerable class ($1, 2, 3 \ldots$). For Cantor proves that the continuum is not denumerable, that is, it cannot be equivalent to one of its parts, which is made to correspond to the series of natural numbers. On the other hand, the class of points of a segment can be put into a one-to-one correspondence to the class of points of a square (despite the apparent wider extension of the latter); they have, therefore, an equal power.

Cantor's construction does not stop here. Beside the extension of cardinal numbers he also examined the extension of ordinal numbers. Owing to his profound analysis of the character of the *order* of the natural series $1, 2, 3 \ldots$, he was able to define *transfinite ordinals* with respect to *well-ordered* series

$$1, 2, 3 \ldots \omega, \omega + 1 \ldots$$

which are subject to the requirement that every sub-class have a first term. It is especially this circumstance that

gives rise to the synthetic significance of the *principle of mathematical induction,* which exactly excludes well-ordered series containing transfinite terms, and which thus characterizes infinite series *of the lowest order.*

We cannot dwell any longer upon this beautiful theory. It is sufficient for our purpose to have shown that the Cantorian analysis of the infinite has brought to proper light the relative significance of the fundamental axiom of inequality. Despite the evidence of this axiom, it appears to us to-day not as a logical judgment, but as a property characteristic of the concept of the "finite" in contradistinction to that of the "infinite."

We have only to add a remark bearing on the limits of the theory of aggregates. If we ascend the ladder of infinites, trying to reach so to speak the maximal, either cardinal or ordinal infinites, we shall meet with new and, as it seems, insuperable contradictions. This happens in case we wish to consider the series of all possible transfinite numbers. For as soon as this is taken as a whole, it can be made to correspond to an ultimate transfinite Ω, and when we have Ω it will be possible to introduce a successor $\Omega + 1$ into the series (the paradox of Burali-Forti). Equally contradictory is the concept of the "assemblage of all classes which are not members of themselves." For it can be shown that this assemblage—considered as a class—both is and is not a member of itself (Russell's paradox). These antinomies give rise to various interpretations, reviving the controversy between realists and nominalists, as described by Du Bois-Reymond. From our point of view they bear witness to the illegitimacy of the suppositions called forth in this field by a realistic conception. They can be solved in the

light of our fundamental conception that it is a question here always of mental constructions only.[1]

26. The Logical Form of Postulates in the Recent Criticism of the Principles of Geometry

It is not sufficient to recognize that the evidence of principles implies presuppositions of a synthetic and objective character. Criticism has to take a new and perhaps more difficult step. It has to learn to give *logical form* to the postulates which are assumed as a basis for a deductive theory. The full significance of this requirement shows itself in geometry better than in arithmetic. It is all the more important to dwell at some length upon this, as there still are eminent mathematicians who have not grasped its value.

Let us imagine a geometrician who has taken cognizance of the work of Riemann and Helmholz or of that of Klein [2] on projective geometry. If he wishes to use the utmost vigor in laying down the foundations of his science, he will have to have a precise idea of the structure of a deductive theory. What criteria will he have to assume?

[1] Cf. B. Russell, "On Some Difficulties . . ." (*Proc. of the London Math. Society*, 1906); "Les paradoxes de la logique" (*Revue de Métaphysique*, 1906, cf. ibid., 1910, 1911).

H. Poincaré, *Science et Méthode* (ch. IV, V). Translated into English by Francis Maitland as *Science and Method* (N. Y., 1915).

L. Brunschvicg, *Les étapes de la philosophie mathématique*, Paris, 1912 (Ch. XVII).

F. Enriques, "Sur quelques difficultés soulevées par l'infini mathématique." ("Actes du Congrès de philosophie mathématiques de Paris, 1914," published in the *Revue de Métaphysique*.)

[2] Ample references on these subjects can be found in art. III, A. I of F. Enriques: "Prinzipien der Geometrie" in the *Enzyklopaedie der mat. Wissenschaften* (1907) (French translation, 1911).

History does not have to make any guesses on this score. The most natural means for getting an insight into the results of such investigations is furnished by the works of the French critics Duhamel [1] and Hoüel [2] who, generally speaking, represent a mode of thinking characteristic of the teachers of our generation.

Duhamel accepts the Cartesian criterion of evidence as the only means by which the truth and falsity of our judgments can be established. Although he admits that the feeling of evidence is not infallible, he believes, nevertheless, that there are truths which are evident to all minds, and that these have to form the starting point for scientific development. The purpose of deductive methods is to help us to discover other truths depending upon and participating in the certainty of the evident truths (op. cit., 15 ff.).

"The definition of a thing is the expression of its relations to other known things." It follows necessarily from this relative conception of definition that there are *non-defined* things which are admitted owing to a feeling of evidence (pp. 16-17). A science of reasoning will be, from this point of view, the sum total of the consequences which necessarily follow from the data admitted in regard to a thing. The thing of course must be well known either by definition (by reducing it to other more known things) or by the admission of certain evident properties which are sufficient for the exact determination of the thing and hence of all its laws (p. 29).

Duhamel recognizes that a system of principles (axioms or postulates) takes the place of definitions of

[1] *Des méthodes dans les sciences de raisonnement*, 2nd ed., 1875.
[2] *Essai critique sur les principes fondamentaux de la géométrie*, 1867.

primitive non-defined concepts in a deductive théory. The sharp criticism to which he subjects Legendre and Euclid in the second part of his work (*Science de l'étendue*) does not, however, entitle us to suppose that he took into account the requirement of using a logical form in the enunciation of the principles. Whether these principles are the product of an imaginative vision or of an intuitive evidence, they remain essentially for him an experiment performed in the imagination and described by them.

This conception comes out clearly in the principles upon which Hoüel proposes to build geometry. After defining *figure* as "an assemblage of points, lines, and surfaces regarded as invariable in form," he enunciates the four following fundamental propositions derived from experience:

1) *Three* points are sufficient to establish the position of a figure in space.

2) There exists a line (straight line) whose position in space is determined by any *two* of its points and it is of such a nature that any part of it can be exactly placed upon any other part, provided that these two parts have two points in common.

3) There exists a surface (plane) such that a straight line passing through any *two* of its points is contained there entirely, and such that any of its parts can be exactly placed upon the surface itself, either directly or after having been turned over.

4) Through a given point only one straight line can be drawn parallel to another given straight line.

And this way of founding geometry seemed so important to Betti and Brioschi that they found it necessary

to reproduce these principles in their edition of Euclid's *Elements!*

At this point the lay reader will naturally stop. It is clear that Hoüel's postulates enunciate facts or properties of the entities studied in the form of an appeal to intuition or to an experiment performed in the imagination. But is this not the proper character which they have to possess as synthetic judgments? And what other form could they assume?

In order to understand this we have to recall to our mind the concept of abstract geometry. If primitive concepts A, B, C . . . are given, a postulate introduces among them a certain relation:

$$\varphi \ (A, \ B, \ C, \ . \ . \ . \ )$$

We try then to translate this relation, by asking ourselves whether it is true or false for other interpretations of A, B, C, . . . In general *the translation has no sense if this relation makes a direct appeal to the intuitive meaning of* A, B, C, . . . How shall we translate, for instance, Hoüel's principles if we are going to abstract from the special meaning given to "motion" understood as a physical operation performed upon figures?

The *logical form* to be given to postulates is precisely that of *relations having a meaning independent of the particular content of the concepts,* that is, of such general relations that they can obtain among "abstract entities."

The first work to give the principles of geometry (although regarded as having an empirical content) the form

of purely logical relations is the *Vorlesungen ueber neuere Geometrie* (Leipzig, 1882) by Moritz Pasch.

In this work we meet, for the first time, with a full realization of the demands which the logical form of a deductive theory has to satisfy. This statement retains its value even when it is maintained that—without going back to Frege's *Grundlagen* of 1884—H. Grassmann's (1844) and Ch. Peirce's critical systematizations of the foundations of arithmetic already satisfy this logical condition. For we cannot ascribe to these works the decisive value possessed by Pasch's criticism, since the axioms of arithmetic appear by themselves as logical relations. And what we have said in regard to Duhamel and Hoüel illustrates our point.

At most one can admit that the arithmetical theory of Grassmann and Peirce—taken over and expressed in symbols by Peano in 1889—helped Peano to understand the significance of Pasch's important innovation, a thing that seems to have actually escaped the attention of other geometricians before 1889. This is true even if, on the other hand, the influence of the latter's geometrical treatise cannot be shown in the *Arithmetices Principia*.[1]

However the case may be, Giusseppe Peano translated Pasch's Principles (with a few slight modifications) into the symbols of mathematical logic [2] in 1889 after he had embarked in the preceding year upon a thorough analysis of logical calculus. But it was exactly on account of the symbolic form, which makes reading by no means pleas-

[1] The theory of ordinal numbers introduces here three primitive undefined concepts besides those expressing logical relations, and for this reason justifies criticism in several respects.

[2] *I principi di Geometria logicamente esposti* (Turin, 1889).

ant, that the influence of Peano's work could not make itself felt at once and that it spread gradually only later.

As far as we can judge from our own recollections, we may say that the sense of the logical form had to be regained as a personal discovery by each of the critical mathematicians that belonged to the same generation. Of course, the possibility of a general and more or less direct influence of the predecessors cannot be absolutely excluded.

The acquisition of this "logical sense" can be seen distinctly, despite certain obscurities of thought, in Giuseppe Veronese's work, *Fondamenti di geometria* (Padova, 1891), and then in our own researches on the foundations of projective geometry (1894).[1] Some notes by Giovanni Vailati [2] and by Mario Pieri,[3] which attach themselves to these researches and which use Peano's symbols, are sufficient to show that the nature of the required logical form was conceived essentially in the same way, although it was given different expressions. The same thing can be said also in regard to the critical notes of Alessandro Padoa on the work of Veronese.

While these ideas were spreading in Italy, David Hilbert arrived at them independently in Germany. His work on the principles of geometry (where problems of high mathematical value are solved) was begun in 1899 with the *Grundlagen der Geometrie*.[4] How little the logical criteria adopted in that work were at that time diffused among mathematicians can be seen from the fact that the logical form of Hilbert's principles called

[1] *Rendic. Instituto Lombardo,* 1894.

[2] *Rivista di Matematiche,* 1895.

[3] *Atti dell' Accademia di Torino,* no. 3, p. 607 (1895).

[4] 3rd ed., Leipzig, 1909.

forth the admiration and even astonishment of Henri Poincaré.

From this moment on the sense of the "logical relation" and the corresponding requirement in regard to the enunciation of principles seem to have become definitely acquired by the consciousness of the mathematical public. Although its formula is seldom stated explicitly, various examples as well as numerous critical and polemical works are sufficient for orienting the minds of a new generation. The following works have especially contributed to this diffusion:

The writings of the pupils of Peano (Vailati, Vacca, Padoa, Pieri . . .) and particularly the *Formulario Matematico* in five editions (1894-1906), which is a collective work of the school;

The researches of both the German and American pupils of Hilbert; some of these (especially those of Max Dehn) deal with problems of great mathematical importance;

The collection of the *Questioni riguardanti la geometria elementare* edited by Enriques with the collaboration of various mathematicians. It was published in 1900 and translated with numerous additions into German in 1900-1919. The second Italian edition appeared in a largely extended form (vol. I, 1912) and under the new title of *Questioni riguardanti le matematiche elementari;*

Finally also the text-books of elementary mathematics published both in Italy and in America (for example, by Halsted) with the purpose of introducing the new logical point of view into the teaching of mathematics, to be sure, in a form and to an extent that are compatible with didactic criteria.

27. EXAMPLES OF PASCH'S LOGICAL ANALYSIS

The non-specialists will be grateful to us for our attempt to explain briefly with the help of some examples the value of that logical form of principles which we have defined above abstractly. We shall disregard details and things which are not essential for our purpose.

It is a question here of examining the postulates of plane geometry, which Pasch bases on the primitive concepts of "point," "rectilinear segment" (hence straight line), and "plane surface" (hence unlimited plane). For the sake of simplicity we shall assume as given the concepts of "straight line" and "plane" as well as that of "natural order" (also as primitive) for the points of a straight line from which we derive the definition of a "segment" as an assemblage of points contained between two given points. It will be easily granted that the following two postulates of plane geometry can be accepted as evident: (1) "Two arbitrary points of a plane are joined by a determinate straight line which lies entirely within the plane"; (2) "A straight line drawn in a plane divides it into two parts."

Let us analyze these statements.

In the first we find, beside the non-defined primitive ideas of "point," "straight line," and "plane," also other ideas represented by the words "two," "arbitrary," "joined," "of," "determinate," "lies," etc. Some of these words are due to the accidental character of the grammatical form of the phrase, or are to be regarded as superfluous. The words "arbitrary," "entirely," for instance, are only explicative terms which can be removed by paraphrasing our proposition in the following way: *Hypoth-*

eses, let A, B be points of the plane; *thesis*, there exists a determinate straight line joining A and B which lies in the plane.

Certain words of our proposition such as "two" are seen immediately to possess logical significance; other words can be easily interpreted as expressing logical relations, provided we admit that:

> a straight line is a class of points,
> a plane is a class of points.

The expression "point *of* a plane" or *"in* a plane" signifies then that it is a question here of a point forming one of the members of the class "plane."

But the words "join," "lie," etc., can also be deprived, in the light of these assumptions, of every intuitive particular content, by regarding them as denoting only the logical relation of membership of points in the class "straight line" and of straight line in the more comprehensive class "plane." In short, postulate (1) turns out to appear as a pure logical relation among the concepts "point," "straight line," and "plane," where straight line and point are conceived as classes of points. Since the word "class" designates any group of objects, we no longer see the points, straight lines, and the planes of intuitive geometry but any objects to which the name "points" is given and particular classes of these denoted as "straight lines" and "planes." On account of the arbitrariness of such classes it will be necessary to postulate that they truly contain, just like the intuitive straight line and plane, any number of members and then also that the plane contains "points" not belonging to a straight line. With this necessary proviso, postulate (1) means first

that two "points" contained in a class "plane" are also contained in a class "straight line," and namely in one only, and then that the members "points" of this straight line are also members of the plane. The additional postulate that "the plane contains points not belonging to one of its straight lines" is tantamount to saying that the class "plane" is not identical with the class "straight line" but is more comprehensive.

Let us now examine postulate (2). What does it mean to say that "a straight line divides a plane into two parts"? The sense of our statement is clear if it is a question here of the intuition of a drawn plane. The postulate signifies in this case that a line drawn between two points A and B situated on the opposite sides of our straight line, such as the rectilinear segment AB, has to cross our straight line, that is, to meet it at least in one point. Now this meaning is closely connected with the intuition to which we have appealed here. If we give an abstract sense to the word "part," we shall have a proposition of the following type: we have established a criterion for distributing the points of a plane into two classes (the two parts) with respect to a given straight line of the plane.[1] What does this criterion imply? Shall we say that we have to add a new concept (of an intuitive and experimental origin) to the primitive concepts "point," "segment," "plane"? Or do we have here a concept that can be *defined* in terms of the latter?

[1] To give an idea of the possible arbitrariness of such a criterion, we may remark that we can obtain two parts of a plane with respect to a straight line "r" by distinguishing the points whose distances from "r" are greater or smaller than a given length, or also (if we prefer a less intuitive division) by distinguishing the points whose distances from "r" are or are not commensurable with a certain unit.

The two parts into which a plane is divided by one of
its straight lines can be indeed defined as follows: We
shall say that two points A and B of a plane, not belong-
ing to one of its straight lines *r, are on the same side* of *r,*
if the segment AB has no point in common with the
straight line, and, on the other hand, that two points A
and B *are on* the opposite sides of *r,* if the segment AB
has one point in common with *r.* This definition obviously
contains nothing but the above mentioned primitive con-
cepts together with their logical relations as subordinate
classes. It permits us to define two parts of a plane with
respect to a straight line *r* and *to a point* A chosen out-
side *r.* Is it legitimate to maintain that such a division is
independent of the choice of the point A?

It is here that we see the true significance of postulate
(2), which exactly asserts such an independence; this is
enunciated in an explicit form by Pasch. Let the follow-
ing two propositions be given: "Two points B and C
(outside *r*), which are on the opposite sides of *r* with re-
spect to A, have to be both on the same side"; and "a
point C, which is on the opposite side of *r* with respect
to A, has to be on the opposite side also with respect to
B when B is on the same side as A." If these propositions
are valid, then the following *postulate of Pasch* must
obtain: If a straight line *r* contains none of the points
A, B, C but lies in their plane, and if it has one point in
common with one of the three segments AB, BC, CA,
that is, if it passes through one of these segments, it
also passes through one of the other two segments but
not through the third.

In our customary geometrical language we shall say:
If a straight line lies in the plane of a triangle but

does not pass through any of its vertices, and if it passes through one of the sides of the triangle, it also passes through one and only one of the other two sides.

In this way we have made explicit what was contained in postulate (2) in a covert form and we have shown what it properly means to divide a plane by one of its straight lines. It is clear that this meaning can be transferred to any abstract interpretation of our primitive concepts. We can find interpretations (for "point," "segment," "plane," "straight line") which may render this meaning true or false. We cannot, however, find such interpretations which would render *meaningless* the division of the class "plane" by the subordinate class "straight line" and with respect to a member "point." This is impossible at least so long as we retain the postulates of connection owing to which "two points determine a segment," etc.

The principles of equality or of superposition of figures through motion (congruence), which, as we have seen, were enunciated by Hoüel, have also to be submitted to a similar logical analysis. It will be then brought out that the concept "motion" implies the idea of a particular "correspondence" or "transformation" of space, that is, of a class of points; this idea (susceptible of various interpretations) will turn out to be purely logical. These principles will thus have to be enunciated either by postulating the properties of the relationship introduced in this way among congruent figures (the relationship itself being regarded as primitive), or by assuming as primitive the concept of "transformation-motion," and

by postulating the properties of a system of transformations. For instance, the transitive character of congruence expresses itself in the property possessed by a system of motions of forming a *group of transformations* (Sophus Lie). This means that the transformation obtained by performing two motions consecutively or, in other words, by forming their product, is again a motion, that is, it belongs to the system.

28. LOGICAL OPERATIONS: SYMBOLICAL AND PSYCHO-LOGICAL ANALYSIS

The development of the ideas expounded above naturally leads to a criticism of "logical relations." A definition of these relations can be said to be virtually contained in the classical analysis of propositions, where a distinction is made between a "subject" and "predicate" connected by a "copula." But the subtle disquisitions of scholastic language and especially the paradoxical conventions introduced from time to time into the formulations of mathematical theories, show how insufficient and inaccurate ordinary language is from the point of view of a complete analysis of thought. Hence the idea of replacing verbal analysis by symbolical analysis and of shaping accordingly a new language after the fashion of algebra.

We have shown in § 15 the presence of this idea in the *Characteristica universalis* of Leibniz and have traced its sources. We have also pointed out in § 19 the transformations it underwent in the Leibnizian school (Lambert, Segner . . .). But the same idea appears in an entirely independent way in the works of the English mathematical logicians of the nineteenth century: Boole,

De Morgan, Peirce. . . . Through Schröeder [1] the works of the English logicians lead to the most recent attempts which regain contact with Leibniz in an interesting manner.

It is not our intention to present here a detailed account of symbolic logic. What we wish to do is to give a brief summary, to indicate the manner in which this logic has tried to explain the proper meaning of logical relations and to compare this attempt with a direct examination of the processes of thought.

The fact that the construction of a symbolic language is to be met with twice in thinkers who had no knowledge of previous attempts in this direction, naturally makes a comparison of the two schools important. As far as the symbols invented by these logicians are con-

[1] We give here brief bibliographical references.

A. De Morgan, *Formal Logic or the Calculus of Inference Necessary and Probable,* 1847;

The article "Logic" in the *Encyclopedia Britannica,* 1860;

Cambridge Phil. Transact., vols. VII, VIII, IX, X. Cf. especially in the last vol., "On the Syllogism and the Logic of Relations," 1860;

G. Boole, 1, *The Mathematical Analysis of Logic,* 1847; 2, *An Investigation of the Laws of Thought,* 1854;

W. S. Jevons, 1, *Pure Logic . . .,* 1864; 2, "On the Mechanical Performance of Logical Inference," *Phil. Trans.,* 1870; 3, *The Principles of Science, a Treatise . . .,* 2nd ed., 1877;

C. S. Peirce, "Three Papers on Logic." *Proc. of the Amer. Acad. of Science,* 1866-70. Cf. especially the third paper, "Notations for the Logic of Relatives";

H. MacColl, *The Calculus of Equivalent Statements,* 1878;

J. Venn, *Symbolic Logic,* 1881, 2nd ed., 1894 (ample historical and bibliographical references) ;

E. Schröder, *Operationskreis des Logickalkulus,* 1877;

Vorlesungen ueber die Algebra der Logik, 1890, 1891, 1895.

An excellent and up-to-date critical account of the whole field of symbolic logic is offered by Clarence I. Lewis in his *Survey of Symbolic Logic.* University of California Press, 1918 (Note of the translator).

cerned one can detect an essential similarity.[1] It is, however, the concept of symbolic analysis itself that reveals a wide philosophical difference, to which we have already referred incidentally above (§ 19) and upon which we wish to throw more light here. This difference can be explained by applying—in the widest sense—to the two schools the names of the two contending parties of medieval Scholasticism: *realism* and *nominalism*.

The realistic presupposition according to which logic corresponds to a natural classification of entities finds its expression in Leibniz, as we have already seen (§ 15), although this classification was meant by him to determine not the existent but only the possible. This presupposition reveals, on the other hand, a mental disposition prone to proceed *deductively*, from the general to the particular, and, consequently, a certain preference for regarding concepts *intensionally*, that is, as an assemblage of attributes or properties, *rather than extensionally*, that is, as an assemblage of objects.

This tendency has, however, not been worked out in a uniform manner. The symbolic notation of Leibniz and his followers indeed presents some oscillations. Leibniz, for instance, defines equality between concepts extensionally. Thus,

$$a = b$$

means for him that the two classes denoted by a and b include the same objects, although the two concepts are defined by means of different properties. But when he introduces the logical sum $a + b$, he interprets the nota-

[1] Cf. Venn, op. cit. (Ch. XX).

tion intensionally: $a + b$ is the more complex concept
to which belong the properties or attributes of both a
and b [1]; (not the class formed by the objects a and b,
as an extensional interpretation implies).

In any case, what above all entitles us to see in Leibniz
a representative of a realistic and intensional conception
of logic is his search for *simple ideas* from the combina-
tion of which (as we have seen in § 15) all possible con-
cepts were to result. And it is this endeavor that bestows
its true importance upon the construction of symbolic
logic as an inventive art. We find here a revival, in a
clearer form, of the alchemistic idea expressed in the *Ars
magna* of the Catalan mystic Raymond Lully (1235-
1315). By arbitrarily arranging material or formal con-
cepts upon three circles revolving around a point, he was
able to obtain at will all their possible combinations. A
peculiar mixture of truth and fancy! And who knows to
what extent this idea still survives in the faith which
recent mathematical logicians bestow upon the creative
magical or heuristic virtue of symbols?

But the English logicians, Boole and De Morgan, are
nominalists, at least after the manner of conceptualism
and terminism. Symbolism indeed is for them a pure in-
strument for the analysis of thought, the process of
thought being regarded by them above all from an
inductive point of view, as proceeding from the particular
to the general. The logic founded by them thus is an ex-
tensional one.

In support of these statements we may point out that
De Morgan follows Condillac not less than Kant. The
formal logic studied by him is an examination of the

[1] *Specimen demonstrandi*, ed. Erdmann, p. 94.

laws of mental action, which are independent of subject matter (cf. § 18). A complete analysis requires an adequate language. The ordinary logical system is incomplete, because it follows faithfully spoken language, without asking itself whether its own limits are properly the limits of thought. De Morgan therefore insists upon a criticism of language, for language contains positive terms without corresponding negatives, and also copulas having different values, being sometimes convertible (as we see in the case of judgments of equality) and sometimes implying a correlative copula (is the father of, the son of). In this way he is led to introduce his symbolic system, which exactly aims to prevent such ambiguities.

George Boole[1] also tries to give a critical explanation of the reasons why ordinary language is not a perfect means for expressing thought. And it is from this point that he proceeds to the most general consideration of symbolic languages. The elements of which all language consists, he says, are signs or symbols. Words are signs. Sometimes they are said to represent things; sometimes the operations by which the mind combines together the simple notions of things into complex concepts.

A sign is an arbitrary mark, having a fixed interpretation and susceptible of combination with other signs in subjection to fixed laws dependent upon their mutual interpretation (op. cit., p. 25).

All the operations of language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements:

[1] *Laws of Thought,* 1854.

1) Literal symbols, as $x, y \ldots$ representing things as subjects of our conceptions;

2) Signs of operation, as $+, -, \times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements;

3) The sign of identity $=$ (p. 27).

The signs of operation—for Boole just as for Leibniz, Lambert, Segner—are derived from algebra. Lambert had borrowed from algebra also the signs $>$ and $<$ for designating the *inclusion* or subordination of two concepts, that is, the fact that one class is contained in another. But the formal laws of these operations reproduce only in part the rules of algebraic calculus. Boole has introduced two other characteristic signs, namely 0 and I:

0 for designating the empty class, namely the logical "null";

I for designating the "universe of discourse," that is, the assemblage of all conceivable objects.

These rules can be used as a means for establishing general rules that are apt to replace long logical calculi, by setting forth the formal solution of the logical equation

$$f(x) = 0.$$

We do not deem it important to enter upon a special examination of Boole's symbolic system. We shall, however, offer a scheme of possible interpretations of algebraic operations (adopted in fact by him and by some of his predecessors), where an interesting duality is brought to light.

EXTENSIVE NOTATION	INTENSIVE NOTATION
$a + b$	$a \times b$
reads: *a* or *b*, combination of the two classes formed by the objects *a* and *b;*	reads: *a* or *b*, concept defined by the common properties of *a* and *b;*
$a \times b$	$a + b$
reads: *a* and *b;* the overlapping of the two classes *a* and *b;*	reads: *a* and *b*, concept defined by the totality of properties belonging to *a* or *b;*
$a > b$	$a < b$
the class *a* includes *b;*	the concept *a* possesses all the properties of *b;*
0	*I*
class empty of objects;	concept to which all properties belong (non-compatible);
I	*0*
universe of discourse, that is, class containing all possible objects.	concept without properties, that is an indeterminate entity.

This shows that the above defined logical operations can be expressed in a double language and that the system of notation based upon the analogous use of the signs $+ \times > < 0\ I$ is susceptible of two correlative interpretations, of an extensional and intensional one. And since the correlative operations possess the same elementary properties, the result is that "the formal properties of logical calculus and hence logical propositions remain invariable when we change the signs $+ \times > < 0\ I$." This is the essence of De Morgan's (1858) and Peirce's (1867) law of *logical duality*.

For instance, the distributive property of logical multiplication with respect to the sum:

$$a \times (b + c) = a \times b + a \times c,$$

becomes the correlative dual

$$a + b \times c = (a + b)\,(a + c),$$

which (keeping in mind the law of simplification $a^2 = I$) expresses the distributive property of logical addition with respect to the product.

The preceding synthetic summary clearly shows the use which can be made of algebraic symbolism for the analysis of the elementary operations of thought. But Boole's tendency to form in this way an art of calculus or research has practically found no followers. Symbolic logic has become in the last phase of its development rather an instrument for the critical analysis of the principles of mathematics. It is precisely on account of the tendency to write entire mathematical theories in symbols that it becomes necessary to change algebraic signs into entirely new ones (a change of pure form which will be sufficient merely to indicate) as well as to introduce other signs whose formal properties have to be defined exactly.

This end was achieved almost simultaneously and independently through the construction of two ideographic languages, of which one was due to Gottlob Frege and the other to Giuseppe Peano.

Frege's starting point was the question whether the principles of arithmetic rest upon empirical data or upon an exclusively logical basis. The results of his inquiries he embodied in his *Begriffsschrift* of 1879. We may

remark here that at that time Frege had no knowledge of the works of his predecessors, except of those of Leibniz.[1] He pursued the same critical end in his *Grundlagen der Arithmetik,* published in 1884. There he examines the views of various mathematicians and philosophers in regard to the concept of number and he concludes by advocating the necessity of using an ideographic language for representing the principles of arithmetic. This task was finally accomplished by him in the *Grundsaetze der Arithmetik begriffsschriftlich abgeleitet* (Jena, 1893, 1903), where the original symbolic system is considerably modified.

Peano began to occupy himself with symbolic logic—in close connection with Schröder—in the introduction to the *Calcolo geometrico secondo l'Ausdehnungslehre di Grassmann, preceduto dalle operazione della logica deduttiva* (Turin, 1888). The following year he presented a complete symbolic treatment of the theory of numbers in his *Arithmetices principia novo methodo exposita.* Although the analysis of Dedekind [2] proved helpful to him here, he reproduced above all the concept of H. Grassmann and C. Peirce,[3] from whom he derived the systematic use of complete induction as a means for establishing the properties of operations. In the same year he published *I principi di geometria logicamente esposti,* which offer, with certain formal simplifications, a symbolic translation of Pasch's geometry of position. Peano's system seems to be already fully formed in these works; it undergoes only few and hardly essential modifications

[1] Cf. Jourdain, *Quarterly Journal,* vol. 43, p. 238.

[2] *Was sind und was sollen die Zahlen,* 1888.

[3] *American Journal,* 1878.

in the subsequent expositions.[1] (It is amazing to see the attitude of an almost voluntary ignorance maintained by this school towards the development and the refined critical views of Russell!)

Competent critics, like Russell and Jourdain, have found that Frege's symbols express a much more refined logical analysis than those of Peano. Frege's notations, however, are obscure, whereas Peano's ideographic language satisfies the economical requirements of simplicity. It is for this reason that mathematical logicians have adopted it by preference. Our rapid survey will also confine itself to this system.

Both Peano and Frege start in their logical analysis, unlike Boole, not from the calculus of classes but from the calculus of judgments and propositions. As a matter of fact Boole (and Lambert before him) already had averred that the signs of operations and relations among classes, while retaining their fundamental properties, are capable of being interpreted as relations among and operations upon propositions. We may also remark

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[1] The principal expositions to which we refer here are to be found in the *Rivista di Matematica,* vol. I, Turin, 1891 (after vol. VI the title becomes *Revue de Mathématiques,* and from vol. VIII: *Revista de mathematica*) as well as in the introductions to the *Formulaire de Mathématiques* in five editions: 1894, 1897, 1899, 1902, 1905 (the last ed. under the title *Formulario Matematico*). Among the works of Peano's school the following ones deserve to be mentioned:

G. Vailati, *Scritti,* Florence, 1911. The articles dealing with mathematical logic and its history have the numbers 1, 2, 4, 5 (1891-94), 27, 39 (1898-9), 88 (1901), 102 (1903), 136, 137 (1905), 197 (1908);

Burali-Forti, *Logica matematica,* Milan, Hoepli, 1894, 2nd ed., 1919;

A. Padoa, "Essai d'une théorie algébrique des nombres entiers précédée d'une introduction logique à une théorie déductive quelconque." ("Actes du Congrès Scient. de Philosophie, Paris," 1900);

"La logique déductive dans sa dernière phase de développement" (*Revue de Métaphysique,* 1912).

that MacColl (1877) had attempted independently an exposition of symbolic logic based upon this second interpretation.

Peano starts from propositions: he writes $a \, \varepsilon \, b$ in order to designate the elementary proposition "a is b" or rather "a is a b", that is, the *membership* of an individual a in the class b. Besides, he interprets the sign — as negative and in the following way the signs $+ \times >$ obtaining between propositions:

$p + q =$ assertion of p or q,

$p \times q =$ simultaneous assertion of p and q

$p > q =$ from p we can deduce q (or p implies q).

(These signs are later changed respectively into ⌣ ⌢ ⊃, just as the signs O and I of Boole have become V and Λ , which are assumed to represent—in the logic of propositions—the true and the false or the absurd. This change is due to a desire for avoiding confusion between logical and arithmetical operations in the ideography of arithmetical theories.)

Signs which have been defined with respect to propositions immediately acquire a meaning applicable to classes, thanks to the fact that a propositional function $x \, \epsilon \, a$ defines a class of individuals a that satisfy the proposition. The form

$$x \, \epsilon \quad a + x \, \epsilon \quad b \text{ and hence } a + b$$

will accordingly designate the combination of x's which are a or b; and $a \times b$ will thus represent the overlapping of the two classes. Similarly, the symbol for the implication of two propositions will represent the inclusion of classes.

In this way it becomes possible to translate syste-

matically the calculus of propositions into the calculus of classes and to rediscover the extensional notation explained above. There is only one exception which is connected with the fact that the "implication of a proposition" forms a proposition, whereas the inclusion of a class a in another class b, a $<$ b, forms a "proposition" but not a "class."

We may point out that Peano does not seem to attach great importance to the fact that the calculus of propositions precedes that of classes in the order of exposition. For he inverts the order in the second volume of the *Formulario,* starting exactly from the latter.[1] He, nevertheless, begins the explanation of the signs with the symbol ϵ , assuming as primitive the concept of the elementary proposition: x is an a. It is exactly the introduction of this new sign ϵ that makes possible, in the opinion of Peano and his pupils, a complete analysis of logic and hence a translation of mathematical theories into symbols.[2]

This point is all the more obscure as ordinary language does not seem to draw a sharp distinction between the relation of membership in a class signified by ϵ and the relation of inclusion of two classes, which can be designated, as long as there is no occasion for ambiguity, by the arithmetical sign $<$. In this way it has happened that some logicians (Schröder, for instance) have failed to

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[1] The same order is adopted by Padoa in his exposition of 1912; this becomes necessary for any one who wishes to consider logic from an extensional point of view.

[2] Jourdain explains with great acuteness (loc. cit., *Quarterly,* vol. 43, p. 299) that the practical success of the systems of Peano and Frege is essentially due to the introduction into logic of propositions containing variables.

distinguish between the two signs. Peano tries to meet the difficulty by indicating certain formal properties which differentiate the two, for instance, the fact that the relation designated by ϵ has no transitive character; this is in his opinion responsible for those fallacies in which the copula is taken in the sense of division instead of in that of composition.[1] A clearer answer to this difficulty can be given by showing that the sign ϵ can be defined by means of the signs given above: $=, <, O$ (the sign $<$ being taken in a sense that excludes equality). Indeed, the membership of an individual a in a class b signifies that the class a is included in b as a *null* class, so that the two relations obtain together:

$$a < b, \text{ if } c < a, \ c = O.$$

Without pushing further the examination of the system of Peano, let us see how he meets the demand for defining logical relations. We are told that we can satisfy this demand by representing these relations by means of the symbols:

$$\epsilon, >, <, =, +, \times, -, O, I$$

which satisfy certain formal properties.

[1] Such is the sophism of the apostles, cited by Peano in *Aritmetica generale,* p. 3:

Peter and Paul are apostles,

The apostles are twelve,

Peter and Paul are therefore twelve.

Peano explains this sophism by showing that the copula "are" is taken in the two premises in the sense of division (ϵ) and that it has therefore to be distinguished from the copula figuring, for instance, in "Barbara" which is to be translated by the symbol of inclusion. But in reality it is obvious that the fallacy is due here to the ambiguity of the middle term, which is taken one time as an abstract term and the second time as a class.

We may remark here that the notion of the fallacies of division and composition goes back to Aristotle. (Sophistical Elenchi, ch. IV, cf. ch. XX), although the passages dealing with these are not quite clear.

All this makes us feel that we are far away from the manner of thinking of a Boole, who undertook to analyze thought by means of symbols. This feeling is produced by the whole formalistic treatment which logic receives in Peano's school. The use of ideographic language is taught here after the manner of living languages or stenography, no attempt being made at an explicit and deeper examination of the meaning which the symbols are intended to represent. Is it still legitimate to follow Boole and to see in the symbols expressions of mental operations? In this case it is hard to understand both the aversion manifested by the whole school for every attempt at psychological clarification, and the importance, no longer auxiliary but fundamental, attributed to the symbols themselves as if they were charged with a mystic meaning.

It is from the most recent development of symbolic logic exhibited, for instance, in the work of Bertrand Russell that we get an idea of this new meaning, of which the above mentioned writers seem to have only a vague intuition.

Russell has submitted symbolic logic to an acute and deep criticism (often perhaps too subtle) in a highly philosophical spirit.[1] The thinker who has little sympathy for the practical trend of Peano's exposition will find

[1] B. Russell, "Sur la théorie des rélations" (*Revue de Math.* of Peano, 1902). *The Principles of Mathematics*, vol. I, Cambridge, 1903. B. Russell and A. N. Whitehead, *Principia Mathematica*, 2nd ed., Cambridge, vols. 1-3, 1925-27. B. Russell, *Introduction to Mathematical Philosophy*, New York, 1919. See also numerous articles in *Mind, Proceedings of the London Math. Society, American Journal of Mathematics, Revue de Métaphysique*, etc. Cf. L. Couturat, *Les principles des Mathématiques*, Paris, 1905.

here great reason for satisfaction. Russell develops the presuppositions of Peano's system from a decidedly realistic point of view which leads him through the work of Bolzano and Cantor to a more intimate contact with Leibniz. He comes to the conclusion that a proper understanding of symbolism tends naturally to bring us back to the Aristotelian position in the degree as we break with Boole's psychologism.

The calculus of propositions precedes for him that of classes. But this fact expresses for him a fundamental logical relation and not the contingent character, so to speak, which it assumed in the development of Peano's system (which also manifests a certain realistic tendency). In Russell's system, the notion of proposition is thus a primitive notion and that of class is a derived one. Our philosopher cannot, consequently, be satisfied with Peano's definition of ϵ, which explains the proposition as membership of an individual in a class. On the contrary, he tries to define class by means of the propositional function p_x and he uses the sign $\Rightarrow$ (the inverse of ϵ, employed also by Peano), which signifies, when placed before p_x, "the totality of x's satisfying the given proposition."

But the proposition in its turn is for Russell only a particular case of a more general primitive concept, namely of *relation*. Peirce and Schröder (after De Morgan) had already studied relations which can connect certain couples of objects or terms x, y in various manners. But for them relation is defined extensionally, as the assemblage of all the couples (x, y) that satisfy the relation. Russell rejects this view, because he sees that it is possible to give different meanings to relations having

the same extensional field (the case of concepts defined by different attributes—for instance, equilateral and equiangular triangle—and yet having the same extension). The extensional conception is above all unacceptable to him, because the couples connected by a relation are characterized by *order;* they cannot therefore be classes. For order itself also forms, for Russell, a certain relation among members of a class. Russell thus writes x R y in order to represent in general a relation holding between a certain *domain x* and a certain *converse domain y* which form together the *field* (x, y), where R assumes meaning. In this way he establishes a *logic of relations,* which exactly presents itself as a generalization of the logic of propositions. It is this which forms, in the opinion of Couturat,[1] the most original part of the work of Russell.

It is not our purpose to pursue further this examination. We wish only to find out what significance logical relations have accordingly for Russell. We may say that they express for him the most general relations that can obtain between entities of any possible world. In other words, their proper significance is to be found not in an analysis of our thoughts but in the truths of a metaphysical universe to which all sense existents are subordinated.

Russell explicitly condemns all psychologistic tendencies, and already in the *Principles of Mathematics* he denounces "the totally irrelevant notion of mind." This assertion is often repeated by him. He says, for instance, in an article in the *Hibbert Journal:* [2]

"Throughout logic and mathematics, the existence of

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[1] *Les principes des Mathématiques,* Paris, 1905 (p. 27).

[2] July, 1904, p. 812.

the human mind or any other mind is totally irrelevant; mental processes are studied by means of logic, but the subject-matter of logic does not presuppose mental processes and would be equally true if there were no mental processes. It is true that in that case we should not know logic; but our knowledge must not be confounded with the truths which we know."

The implications of this realism are clearly seen in Russell's metaphysics as it developed through his criticism of the system of Leibniz.[1] In this remarkably acute criticism, in which he attaches himself to the scholastic aspect of the Leibnizian *Monadology,* he almost succeeds in getting at its true historical basis. According to Russell, the philosophy of Leibniz, like any other healthy philosophy, takes its starting point from the analysis of propositions (reviving Aristotelianism in a curious fashion). All logical relations are from this point of view regarded as being reducible to the subject-predicate type. The subjects of these propositions—identified for Leibniz with substances—can receive different predicates in the course of time, but they can never become predicates themselves. From this Leibniz is supposed to infer that all these invariable subjects can never enter into relations among themselves. Hence the dictum: "Monads have no windows." Only the discovery of relations contradicting these premises, namely of those that are irreducible to predicative propositions, pushed Leibniz in the direction of a theory—advocated later by Kant—which ascribes to relations a mental significance. Russell, however, far from proceeding on this road (whatever its origin may be), re-

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[1] *A Critical Exposition of the Philosophy of Leibniz.* . . . Cambridge, 1900. Cf. *The Problems of Philosophy.*

turns to the original scholastic position of Leibniz, and tries to meet the insufficiency of the logical analysis by the introduction of non-predicative relations. Instead of substances in the Leibnizian sense he has, accordingly, a world of relations which are neither sensible nor mental but which form the *world of universals*. Through the conjunction of *simple*, properly scrutinized relations we arrive at an a priori science of all possibilities.[1]

A tendency entirely opposed to the realism of Russell is represented by ourself.[2] For us, just as for Boole—at whose position we have arrived independently—logic is the sum total of the laws governing a *mental process* that can be represented only in a fictitious manner in the static form of symbolism. To explain logical relations thus means to establish the mental operations which they designate. This suggests that logic is a part of psychology. The reader is, however, warned that this is to be taken in a rational sense so as to avoid the objections made by Kant to such a conception (cf. § 19).

The first step in the psychological analysis of logic is to recognize objects or *individuals* which thought regards as invariant. The elementary judgment of *identity* and *difference* between two objects in this way acquires a value that is independent of time. The conditions which

[1] One of the remarkable results of Russell's anti-critical position is his justification of absolute motion as motion with respect to space. *Principles*, ch. LVIII. Russell has since given up this view. (Note of the translator).

[2] *Problems of Science*, ch. III. Cf. also "The Problems of Logic" in the *Encyclopedia of the Philosophical Sciences*, edited by Windelband and Ruge, vol. I (London, 1913).

confront thought in this respect are expressed in the prin-
ciples of logic (laws of identity, contradiction, and ex-
cluded middle). Mind combines logical objects by means
of associative operations. Several objects *a, b, c, . . .*
can be *combined* in a *class* (*a* and *b* and *c . . .*) or
arranged in a *series* (the ordered class *abc . . .*). In this
way we obtain the definition of the *combination* of two
or more classes as well as of the *correspondence* which
can be established (by means of an associative process)
between the members of two classes.

The inversion of the indicated operations gives rise not
only to the *combination* of two classes but also to
abstraction. The abstract concept of the individual mem-
ber of a class (*a* or *b* or *c . . .*) indeed is obtained by an
operation inverse to the one through which *a, b, c . . .*
are combined together in a class. The inverse movement
from the class to its members is naturally not univocal. It
is for this reason that the abstract term is a new object of
thought representing "any one of the combined members
which thought regards as replaceable by any other
(*equal*)."

The *logical relations* among certain concepts(classes,
series, etc.) prove thus to express logical operations
which enable us to construct these concepts on the basis
of objects or individuals which actually occur in thought
or which are only *possible*. This addition is especially
important in view of the fact that the concepts used by
us introduce classes (like the straight line) in which
infinite objects (points) are combined; it thus becomes
impossible for us first to enumerate in thought these
objects one by one and then to combine them together.
A straight line is a class of points only in the sense that

it implies the assumption of infinite possible points, which have to be conceived—by means of an a priori determination—as combined. The concept of *proposition* in particular—considered as primitive in recent theories of symbolic logic—resolves itself into the concept of the above mentioned operations (constructive of classes, etc.) as well as into that of judgments of *identity* and *difference* between objects (members of these classes, etc.).

We have not extended our criticism to a systematic examination of all logical relations. But a comparison with symbolic logic shows easily that our analysis transcends the limits of Boole and that it is also exhaustive. The dynamic conception of logic replacing the static representation which operates by means of symbols restores to the concept of "ordered series" the full right of figuring among logical concepts beside—and in a proper logical sense *before*—that of "class." The latter concept springs in reality, through a process of abstraction, from a comparison of differently ordered series. Russell's difficulties with the concept of order, which led him to introduce relation as a primitive notion, seem to be solely due to the part played by symbolic expression in the logical process.

29. The Hypothetico-Deductive System

We have seen how the reform of logic was prompted by the criticism of the principles of mathematics and especially by Pasch's analysis. The question naturally arises, where does all this finally lead to? What is the new concept of demonstrative science or of the struc-

ture of a deductive theory that is gained by these re-
searches?

We have said that in a logical sense there are no real
but only nominal definitions, and since these have a rela-
tive character, we resort to certain *primitive non-defined*
concepts, which have to be announced explicitly.

In quite a similar manner deduction bases the proposi-
tions of any theory upon certain premises or principles.
The logician has not to worry over the question whether
these principles find their justification in evidence, or
whether they are of an experimental origin, or whether
they are accepted as "hypotheses" with a view to some
further end. In this sense he does not distinguish between
"axioms" and "postulates"; he has only to see to it that
all the postulates are *enunciated* as such and *in a form
of pure logical relations assumed among the primitive
concepts.*

The system of postulates yields the *implicit definition*
of the non-explicitly defined concepts in the same way in
which a system of equations defines its unknown terms
by limiting their field of variability (comparison of Ger-
gonne)

To this one may make the following objections: I find
in geometry the fundamental concepts of "point,"
"straight line," "plane," etc. . . . designated by *a, b,
c* . . . , as well as their logical relations expressed in a
system of postulates. Do you maintain that these *postu-
lates* truly *define a, b, c* . . . ? Is there not a contradic-
tion between such a view and the abstract meaning which
you tried to bestow with so much care upon "logical rela-
tions"? For if it is possible to give your system another
interpretation (in which a "point" becomes, let us say, a

"circle" and a "straight line" "a bundle of circles," etc.), then good-bye to your definitions of concepts! Or are you perhaps willing to relieve definition (qualified nicely as "implicit") from its special task of "defining," that is, from determining the entities under consideration? And our critic will go on cursing the abstract spirit of mathematicians and preaching that salvation can be found, at least as far as the primitive concepts of science are concerned, only in a return to *real definitions*.

But fortunately for our critic we shall not take him at his word, we shall not challenge him to give us this real definition of geometrical entities. He will certainly be unable to give us this, unless he appeals to observations and approximate experiments, which would yield only physical entities—something thus quite different from the objects of rational geometry. And to the preceding objections we may answer as follows.

To what end do we define the fundamental concepts of a deductive science? This is certainly done with a view to the systematic development of the science. And in this sense the real function of concepts consists in their relations, which we have to postulate explicitly. But can a system of postulates be ever complete in this sense? This question can be met in every single case by applying the test of a one-to-one correspondence to two systems of entities which have to satisfy our postulates. Our result will be positive if such a correspondence can be established between the systems and if the properties of the one can be thus translated into the perfectly homologous properties of the other, so that they will appear as *equal* in an abstract sense and from the point of view of the ideas under consideration. For instance, the systems of

postulates which ordinarily characterize projective as well as metric geometry are complete. For these postulates define projective spaces in such a way that a projective correspondence can be established between them, and they define metric spaces so that it is possible to pass from one to the other through a correspondence which retains also the relations of congruence.

Our critic may grant us that a system of postulates is complete in the just mentioned sense. He may, however, say that space, whether in experimental reality or in intuition, is always something more determinate than what we assert to have defined by means of such a system of postulates. We have to take into account not only the relations which points have among themselves, or which they sustain to straight lines and planes, etc., but also relations which are external to the development of geometry, for instance, relations to matter, motion, and so on.

Well, to this we may answer that in the implicit definition of space (or of the concepts involved in it) which we give in geometry we do not claim to go in the determination of our object beyond the sphere of geometry. There is of course nothing to prevent us from extending geometry to a wider system, to mechanics or physics, by introducing other fundamental concepts and by connecting them by means of new postulates with those which are properly geometrical. This extension of a rational logically constructed science is as a matter of fact rendered necessary by any of its applications to the external world. In this case an approximate correspondence is established between fundamental concepts and *real objects* which are *defined* this time—in a sense different

from the logical one—by means of observations and experiments related to possible actions or reflections which are better suggested than expressed by words. For reason has the following peculiar trait. It aims to represent every form of reality by means of a system of abstract concepts and to extend its sway to ever wider—and so to speak, more real—realities by leaving this system open for an indefinite progress.

Let us come back now to our scientific system in which concepts are said to be defined implicitly by means of their logical relations. We may point out that there is no natural order for deductions which determines the choice of *primary* propositions assumed as postulates. There is not even a necessary order for definitions which would make us regard certain concepts rather than others as primitive. Thus the criticism of the principles of geometry has revealed different orders. Here we can, for instance, define a straight line and a plane by means of a sphere (Lobatchevsky) or we can, conversely, subordinate the latter notion (together with the concepts of congruence and motion implied by it) to the prior concepts of straight line and plane.

One might expect that the conversion of logicians to these criteria would result in a simultaneous defeat of the Aristotelian ontologism. We have had, however, already to admit that this defeat is not definitive and that the spirit of realism survives in new forms. Especially as far as primitive concepts are concerned, the Leibnizian idea of establishing a small number of *simple concepts* (the alchemic vision of Raymond Lully) re-

appears in many thinkers and, as we have seen, in recent mathematical logicians. Lambert, who followed Wolf and, hence indirectly, Leibniz in his work on primitive propositions, said that the *Grundlehre* has to begin with simple concepts, for otherwise definition and demonstration would never end.[1] "Simple concepts in the explanation of things, and their names in verbal explanations form the first principle and serve as a basis for the rest" (op. cit., p. 24). But it is more surprising to find the same idea in the first expressions of Peano's thought. In a review of Frege's *Grundsaetze der Arithmetik, begriffschriftlich abgeleitet* [2] Peano expresses himself in this way: "Mathematical logic does not consist of a series of arbitrary conventions which vary with the whims of the author, but of an analysis of ideas and propositions into primitive and derived. And this analysis is unique. . . . The various ideographies which can be invented have to coincide in the end in case they are equally adapted to represent all propositions, except perhaps as far as the form of the signs adopted are concerned."

It seems that Peano has changed his opinion in regard to this point, since he writes in the *Formulaire de Mathématiques* (vol. II, 1897): "But the differentiation of ideas into primitive and derived is somewhat arbitrary; for if *b* is defined by means of *a* and *a* is defined by *b*, either *a* or *b* can be taken as a primitive idea" (p. 27). We have expounded, he adds, the reduction which seems to

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[1] *Anlage zur Architektonik oder Theorie des Einfachen und des Erstens in der philosophischen und mathematischen Erkenntnis*, Riga, 1771 (p. 19).

[2] *Rivista di Matematiche*, Turin, vol. V, 1895 (p. 123). Cf. Frege's letter of reply, dated Sept., 1896, *Revue de Mathématiques*, vol. III, p. 53.

us to be the most simple, but we are introducing the symbol [D*f*] in order to designate possible definitions in cases where the system of primitive propositions changes.

A frank admission of the logical arbitrariness presented by the choice of a given deductive order is to be found in Vailati. The following statement is characteristic in this respect: [1] "Instead of conceiving the difference between postulates and other propositions as consisting in the fact that the former possess a certain character which renders them 'by themselves' more acceptable, more evident, less debatable, mathematical logicians see in them propositions like all other propositions whose choice is determined by the end pursued by the exposition.

". . . The relations between postulates and the propositions depending upon them could be formerly compared with those which obtain in an autocratic or aristocratic state between the monarch or the privileged class and the other parts of society. The work of the mathematical logicians, on the other hand, has become similar to that of the founders of a constitutional or democratic régime, in which the choice or election of the heads of the government depends, at least ideally, upon their recognized ability to exercise temporally certain functions in the interest of the public.

"The postulates had to give up that species of 'divine right' which they arrogated to themselves in virtue of their pretended evidence; they had to descend from their rôle as 'arbiters' to that of 'servi servorum,' of simple 'employees' of the great 'associations' which form the various branches of mathematics."

We are sure that we are not doing any injustice to the

[1] *Scritti,* p. 690 (1906).

thought of the author—expressed clearly by him on many occasions—by substituting in our quotation the words "postulates" and "propositions depending upon them" by the words "primitive concepts" and "concepts defined by their means."

Once we realize that every scientific theory can be reduced (to use the expression of Pieri) to an *hypothetico-deductive system* whose principles are more or less arbitrary, we commit ourselves to a choice of fundamental concepts and postulates tending to become more and more free. We see this in the case of Peano's school, where the determining motive is logical criticism, as well as in the case of Hilbert's school, where the end in view is a higher mathematical interest. It is interesting to notice that it is exactly in the school of symbolic logic that the search for *simple* concepts has yielded place to the advocacy of a criterion according to which the concepts to be assumed may be very complicated but their number must be as small as possible. This is why the system of Pieri, which has reduced the number of primitive geometrical concepts and of their typographical signs to *two* (whereas the arithmetical system of Peano has *three*), is proclaimed by these logicians as a great progress. It does not seem to have occurred to any of these thinkers that a reduction of this nature is in itself of slight importance. Would any mathematician think that he is achieving greater perfection in the definition of the domain of rational numbers [$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$] by substituting for the three irrational quantities a single complex one formed from them?

But these and other similar objections that can be made to this school in regard to the application of its ideas to the field of mathematics are ultimately due to the fact that this school refuses to submit logical arbitrariness to extra-logical criteria. In this way the exaggerations or errors have an illustrative value for logic in every instance. This is the case with the above described transition from the search for simpler concepts to the insistence upon more complex ones admitting of the greatest numerical reduction. For the pure logician there is no middle way; simplicity, like any other criterion, loses all value in his eyes as soon as it becomes clear that it has no absolute significance. We are thus reminded of Manzoni's mob: once it has become persuaded that some one does not deserve to be hanged, not much energy is required to convince it that the given person has to be carried away in triumph.

30. Independence and Compatibility of Principles

The attempts made at the end of the eighteenth and at the beginning of the nineteenth century to prove the Euclidean postulate of parallels called forth the problem "whether and how it is possible to establish that a proposition or hypothesis is independent of other propositions assumed as postulates." The problem has been solved by non-Euclidean geometry. This geometry shows that the postulate of parallels does not follow as a consequence from any system of postulates characterizing the ordinary concepts of "straight line," "plane," "congruence," etc., so as to yield the premises for the first 27 propositions of

Euclid.[1] For it is in the immediately following 28th proposition where use is made of our postulate.

In what way does non-Euclidean geometry reach this paradoxical result of "proving the impossibility of proving"?

We prove that the proposition x cannot be deduced as a consequence from a system of premises $a, b, c \ldots$, by showing the consistency or compatibility of a system of hypotheses $a, b, c \ldots$, non-x.

What we arrive at in the present case is precisely this. We designate by x the Euclidean hypothesis of the parallels and by $a, b, c \ldots$, the postulates assumed. The hypothetico-deductive system which has for its basis $(a, b, c \ldots$, non-$x)$ can be then interpreted abstractly as relative to the entities of ordinary Euclidean space. Every eventual contradiction that may appear in the course of the development of the system $(a, b, c \ldots$, non-$x)$ will therefore have also to affect the development of the system $(a, b, c \ldots x)$. The hypothesis which makes x depend upon $a, b, c \ldots$ reveals itself thus as absurd as soon as we admit (and this is a necessary presupposition) that the system $(a, b, c \ldots)$ is compatible.

The question of the independence of postulates in this way reduces itself to that of compatibility—a possibility that might not have suggested itself at least in the case of the evident principles of geometry.

But how shall we solve logically the new problem concerning the compatibility of a system of hypotheses?

It is a question here in reality of making sure that the

[1] A simple exposition of this independence will be found in the already mentioned *Conferenze sulla geometria non euclidea* by F. Enriques, edited by O. Fernandez.

deductive development starting from given premises will *never* lead to a contradiction. We have seen how Leibniz and Saccheri grappled with this problem and how they tried to solve it by resorting to *simple ideas,* for they believed on a priori grounds that simple ideas cannot be mutually contradictory and that only complex ideas can contain contradictions.

Lambert also followed the same views: "Possible is what contains no contradictions," "what is, is possible" (*Anlage zur Architektonik,* p. 16). He remarks that this enables us to find possibility in an a posteriori way through experience, that experiments alone and examples by themselves, however, are not sufficient to show directly how far possibility extends. It is necessary to *postulate* here a priori the possibility of complex concepts. "Contradiction requires more than one piece: there are no contradictions in simple concepts."

Among recent thinkers who occupied themselves with this problem, called forth or stimulated by the development of non-Euclidean criticism, we find two tendencies. According to one tendency the compatibility of a system of hypotheses can be proved only a posteriori, in so far as any interpretation is given to the system in the domain of experience. According to the other (following the example of Weierstrass and Kronecker) the abstract meaning of the system is to be reduced to an arithmetical interpretation, where everything is based on the idea of natural numbers. The possibility of this idea is admitted a priori in virtue of the laws of thought.

The empiricistic thesis, advocated especially by Vailati, has to face a serious objection. We may admit with Lambert that "what is, is possible," namely non-contradictory.

In order that we may establish a priori by experience that "something is" in the sense required here, it is necessary that we should be able to translate this experience into perfectly exact judgments containing nothing approximate. But such a requirement is exactly impossible to fulfill, at least in so far as we are dealing with the "continuous," for only experiments referring to the "discrete" can lay claims to exactness. We may of course follow the example of certain mathematical thinkers who rather dislike to give a deeper analysis of the philosophic content of their opinions and who extend experimental reality also to the world of intuition (for instance, geometrical intuition). We may admit, more or less explicitly, that the latter reflects a certain tendency of experience that has been worked out in the course of a long mental evolution. In this case the value of the empiricistic tendency is certainly enhanced. For it will signify then that the compatibility of certain concepts was established not only by the historical development of the science but also by their psychological construction, where we discover the influence of the logical requirement to exclude contradictions.[1]

As to the rationalistic thesis which reduces the demonstration of every form of compatibility to the consistency of the arithmetical system we believe that it indeed expresses the sharpest way of meeting the problem. The basis of such a demonstration is thus made as small as possible, for it reduces itself to the assumption of an infinite series of objects as we find in the natural numbers. This series can be constructed in the light of the simple inner experience of the recurrence of the acts of thought,

[1] Cf. Enriques, *Problems of Science*, ch. IV.

a fact expressing itself in the *principle of mathematical induction*. It is in this sense that we can accept the opinion of Poincaré that the principle of induction yields an existential synthetic judgment to which all other mathematical doctrines can be reduced. On the other hand, all the attempts that were made to prove this principle by deducing it from logical axioms (expressing the laws of association of thought) have turned out to be futile.

The problem of the compatibility of the premises of an hypothetico-deductive system has been clarified by our own psychological analysis of the logical process.[1] The Leibnizian conception of "simple ideas" appears here in a new form, as transferred from intensional (presupposing realism) to extensional logic.

Let us try to submit the meaning of contradictory propositions to a deeper analysis. To this end we have to examine the proper value of the *principles of logic*, of identity, contradiction, and the excluded middle. We meet in the history of thought with two opposed modes of explanation, one of which goes back to the ancient Eleatic philosophy, the other emanates from the Kantian world of ideas. The former regards the principles of logic as properties of the real world, the latter as subjective conditions of thought. The realistic conception of the principles of logic, when taken in a rigorous sense, implies the metaphysical consequences which we find in the system of Plato and the Megarians.

The view which regards the principles of logic as mental laws in reality antecedes Kant. Leibniz already

[1] *Problems of Science,* ch. III.

observed that they seem to spring from the generaliza-
tions of experience; they are, however, presupposed in
every proof and in this way they express the primary con-
ditions of all thought and knowledge. It is true, on the
other hand, that the spread of Kantianism has contributed
to the acceptance of this view. Neo-Kantian logicians,
like W. Hamilton, say clearly that the principles of logic
are fundamental laws determined by the nature of the
thinking subject, that is, they are conditions of what is
conceivable. Stanley Jevons gives a profound discussion
of the question in his principal logical work.[1] Science (he
says in the introduction) springs from the discovery of
identity and difference in the spectacle of the infinite
variety and novelty which nature presents to our senses.
The three principles, which he calls the laws of identity,
contradiction, and duality, express for him "the true
nature and conditions of the discriminating and identify-
ing faculties of the mind." "Are they laws of thought or
of things? Do they belong to mind or to material nature?"
he asks on page 6 of the just mentioned work. He refers
here to the remark of Leibniz quoted above. While he
criticizes John Stuart Mill's psychological and empirical
treatment of logic, he grants Spencer that the principles
of logic can also express objective laws (or most general
facts) adding that it is necessary to distinguish in every
case between the constitution of the mind and the accumu-
lation of knowledge.

Our own conception follows almost the same road in
this respect. The principles of logic express the invariance
of the objects of thought which our will asserts as fixed

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[1] *The Principles of Science.* A Treatise on Logic and Scientific Method
(1873, 2nd ed., 1877).

in the process of reasoning, and which thus serve for the purpose of defining "logical individuals." Now if *a, b, c* are logical individuals which are explicitly presented in thought as distinct, they can never change in the course of any logical operation, they can, namely, never become mixed up so as to make us regard what is different as identical and vice versa; nor is it possible for the new entities (for instance, the classes (ab), (ac), (bc)) created by such operations to become identical and different at the same time. In other words, the logical process can never lead to contradiction when it starts from (a finite number of) individuals asserted by thought. This seems to be in conformity with the Leibnizian conception of "simple ideas," transferred this time to extensional logic. By "simple entities" we here understand no longer the most general concepts, but on the contrary individuals determined by thought, namely objects of any nature whatsoever which are taken abstractly as indecomposable elements of thought. There can be no contradiction involved, for instance, in the fact that thought apprehends several points *a, b, c* . . . at the same time, for the very possibility of comparing these "elements" of thought implies that we recognize them as identical or different and that the result of such a comparison remains unchanged as long as these logical objects continue to be invariable.

In the real logical process we do not, however, operate with *constructed concepts,* where we start from (a finite number of) logical individuals actually established by thought. We deal here instead with *given concepts* whose logical relations presuppose the possibility of a certain construction, namely the representation of these concepts

as "classes," etc., where we start from individuals that can be asserted hypothetically. And since the number of such individuals may be infinite, any actual representation becomes impossible. The question of "the compatibility of the logical relations which we state as implicit definitions of our concepts" exactly refers to the justification of hypothesis which gives the concepts themselves their "assumed existence" as possible products of thought. The principles which enunciate the relations assumed are properly *postulates* in so far as they require the admission of this logical existence, which excludes all contradiction from the resulting deductive development.

In this way we can solve the Hobbesean difficulty in regard to the arbitrariness of mathematical truths. The postulates of an abstract theory, while yielding the implicit definitions of arbitrary concepts, are not themselves arbitrary, since they have to satisfy the condition of the logical existence of the concepts defined, the value of which we have explained above.

We may add a few more remarks to the question of the independence of principles.

If we are dealing with a *system of postulates a, b, c . . .* , we can define either their *absolute independence* or their *ordered independence*. By the latter we mean that *b* cannot be a consequence of *a*, that *c* cannot be a consequence of *a* and *b*, although we do not exclude the possibility of *a* being deduced from *b, c, . . .* This has called forth the following remark on the part of Beppo Levi: [1] Given a system of postulates *a, b, c . . .* , which is characterized by ordered independence, we can

[1] *Memorie dell' Accademia di Torino,* 1904.

always construct a new system of absolutely independent postulates. To this end it is sufficient to substitute the system a, b, c, . . . by the system formed out of the following propositions:

$a^1 = a$; $b^1 = b$ is valid for all the entities which satisfy the condition a; $c^1 = c$ is valid for all the entities satisfying the conditions a and b; etc.

It is hardly necessary to point out that this observation has a purely formal value. In general, when we are dealing practically with an hypothetico-deductive system, we have to require only the ordered independence of the postulates a, b, c, . . . For absolute independence has no meaning in the case of a hierarchy of concepts, where the subsequent concepts appear as specifications of the preceding ones, and where the last postulates thus become subordinated to the concepts figuring in the previous ones. A good example is offered in this respect by the metric-projective systematization of the foundations of geometry. The metric concepts depend here upon the graphic concepts in such a way that it would be meaningless to put some of the postulates of congruence or motion before the postulates of connection obtaining between straight lines and planes or before those expressing their linear or surface properties.

We may also add in regard to the independence of postulates a, b, c that it may be impossible, for instance, to deduce c as a consequence of a and b, although it may be possible to deduce from a and b a *part* of the proposition c. This gives rise to the demand of introducing only simple propositions as postulates, or propositions which cannot be analyzed into the simultaneous assertion of two other propositions which are generally equivalent.

But Padoa, who examined the question of the meaning of simple propositions from an extensional point of view, came to the conclusion that every simple proposition can be reduced to the form

a is different from b,

where a and b are two logical individuals. Every other proposition can be virtually analyzed into similar judgments of difference. For instance, the assertion that a given member belongs to a class implies that this member is different from all the other members which we can regard as being outside the class in question.

The simplicity of postulates, however, has the meaning of an entirely relative norm possessing only an esthetical value. And as to the question about the partial or total independence of a postulate c with respect to other postulates a, b, we have to say that so far it has not been sufficiently studied.

Beside the question of postulates we can put that of the *independence of primitive concepts*. Wherever we find in a given hypothetico-deductive system the concepts A, B, C, . . . which are accepted without definition, we may ask whether one of them cannot be defined (explicitly) by means of the others; for instance, C by means of A and B. But this question has no meaning so long as we do not determine precisely the system of postulates a, b, c . . . by means of which A, B, C are implicitly defined. If this system of postulates is assumed as given, we can prove the independence of the concept C of A and B (that is the impossibility of defining C by means of A and B in the given theory). We find two concrete interpretations for the theory, in which the meaning of

A and B is retained while that of C changes. In this way all the postulates a, b, c . . . are equally satisfied, a certain proposition x, however, turns out to be true for the first interpretation and false for the second. This enables us to define the *irreducibility* of a system of primitive concepts with respect to a system of postulates.[1]

[1] Cf. Padoa, *Essai d'une théorie algébrique* . . . 1900 (loc. cit. no. 16).

FROM INDUCTIVE LOGIC TO THE LOGIC OF SCIENTIFIC SYSTEMS.

In order to grasp the full significance of the concept of the logical structure of scientific theories that was gained in the nineteenth century through the criticism of mathematics, we must take into account the simultaneous and parallel development of the so-called inductive logic. The fate of this logic was also connected with the progress of mathematical sciences taken in their widest sense.

Scientific systems may be approached, on the one hand, from a rigidly formal point of view and analyzed in an hypothetico-deductive manner as devoid of any content. On the other hand, they may be referred to the reality of the observations and experiments suggested by them; they will then be studied with respect to the genesis and verification of hypotheses, that is, from the point of view of the ascending movement of thought. A comparison of these two orders of considerations leads finally to a logic of scientific systems.

31. A. COMTE'S POSITIVISTIC CONCEPTION OF SCIENCE

The logical works aiming at the discovery of scientific truth can be traced back, on the one hand, to Auguste Comte, who published the first volume of the *Cours de philosophie positive* in 1830, and on the other hand, to John Frederic William Herschel, whose treatise *On the*

Study of Natural Philosophy [1] appeared in the same year.

Comte was not concerned so much with the question of method as with the attempt to define, by means of a systematic analysis of the various fields of knowledge, the very concept of scientific explanation. It is well known how he was led in this way to reduce all science to a combination of facts which science describes and sums up in the smallest number of formulas, rigorously eliminating any attempt at going beyond phenomena. "The fundamental character of positivism," he says in the first lecture of his *Cours*, "consists in the fact that it regards all phenomena as subject to natural and invariable *laws*. The accurate discovery and reduction of these laws to the smallest number possible form the end of all our efforts. We have to realize that it is absolutely meaningless and useless to look for so-called *causes*, no matter whether primary or final." This view is immediately illustrated first by the example of the Newtonian gravitation and then by that of the theory of Fourier. He says that general phenomena of the universe are *explained* by Newton's law, as far as it is possible, because, on the one hand, it teaches us to regard the immense variety of astronomical facts as one single and most general fact seen from different angles, on the other hand, this general fact presents itself as an extension of the phenomenon of weight, which is considered as perfectly known only on account of its familiarity. Fourier's theory of heat is referred to by our philosopher especially, because it reveals the emptiness of the controversy between the partisans of a calorific matter and

those who see in heat a state of vibratory motion. It is in the same spirit that Comte explains the general meaning of physics, rejecting the metaphysical hypotheses relative to ethers, fluids, etc. . . . , taken by contemporary mathematical physicists as foundations for their construction.

It would not be amiss to compare with these views those expressed in regard to method by Newton whose explanations offer a model of positivistic philosophy. After pointing out that the *analytic* method has to precede the *synthetic* in the difficult matters of physics, the author of universal gravitation adds:

"The analytic method consists in making experiments and observations, and in drawing general conclusions from them by induction, and admitting of no objections against the conclusions but such as are taken from experiments or other certain truths. For hypotheses are not to be regarded in experimental philosophy. And although the arguing from experiments and observations by induction be no demonstration of general conclusions, yet it is the best way of arguing which the nature of things admits of, and may be looked upon as so much the stronger by how much induction is more general. And if no exception occur, the conclusion may be pronounced as general. But if at any time afterwards any exception shall occur from phenomena, the conclusion shall not be pronounced without these exceptions. By this way of analysis we may proceed from compounds to simple ingredients, and from notions to the forces producing them, and in general from effects to causes, and from particular causes to more general ones until the argument end in the most general. This is the method

of analysis. The synthesis consists in assuming the causes discovered and established as principles, and by them explaining the phenomena proceeding from them and proving the explanations." [1]

Coming back to Comte we find that the value of scientific explanation is quite clearly defined by him in the second lecture of his *Cours*. In his attempt to state the relation existing between science and art he makes prediction the purpose of science. "Science d'où prévoyance, prévoyance d'où action." Comte, however, looks in knowledge largely for the satisfaction of an intellectual need rather than for power. An essential part of this need is for our philosopher the craving for order (an obvious expression of his interest in social and moral questions), which he tries to satisfy by means of a hierarchical classification of the sciences. Although he thinks that the object of all our inquiries is really one, and that we divide it only in order to separate the difficulties so as to solve them better, he does not regard scientific divisions as arbitrary, as some thinkers believe; he grants at most that they are somewhat artificial. It is probably futile to try to reduce the explanations of the phenomena of the universe to a single one (as can be seen from the most rational attempt made in this direction by Laplace, who based himself upon the most general positive law of gravitation); positivistic philosophy is satisfied with the *unity of method*.

Comte's conception of logic is clearly expressed in the first lecture of his *Cours*. Intellectual functions, he says, can be regarded either statically, as determinations of the organic conditions upon which they depend, and

[1] *Optics,* third ed., London, 1721, pp. 380-81.

in this case they become parts of anatomy and physiology, or they can be considered dynamically. "When we regard them dynamically our task is exclusively to study the human mind as it actually proceeds in its operations, and to examine the methods really employed in the different parts of exact knowledge which have already been attained. It is this that forms the general object of positivism. . . ." "We can rise to the knowledge of logical laws only by regarding all scientific theories as so many great logical facts." And after opposing these two roads as the only proper ones to that of illusory psychology, he adds: "Method cannot be studied apart from the researches in which it is employed, otherwise we have only a dead study incapable of fructifying the mind using it," for it can lead to vague generalities only.

"After we lay it down as a logical thesis that all our knowledge has to be based upon observation, that we have to proceed now from facts to principles, and now from principles to facts, and after we indulge in other similar aphorisms, we certainly know much less about method than one who has studied a single positive science more or less seriously, even without any philosophic intention. It is because our psychologists have disregarded this essential fact that they were led to consider their fancies as science. They believed that they had grasped the positivistic method, because they had read the rules of Bacon or the discourse of Descartes." It is true, Comte does not deny that it may be possible later to undertake a priori a true study of method. He believes, however, that such a result could be attained only through a study of the positivistic philosophy, only after we have been led by this philosophy to an exact

knowledge of the general rules that are necessary for the sure pursuit of truth.

Our philosopher seems to have remained essentially faithful to this conception of logic also in the last mystic period of his life, at the time when he wrote the first volume of *La synthèse subjective,* which also remained the last and which has as its sub-title "Système de logique positive ou Traité de philosophie mathématique."[1] We have, however, to point out the importance attributed in this work to feelings, images, which makes him define logic as "the normal collaboration of images, signs, and feelings in suggesting to us the ideas that are suitable to our moral, intellectual, and physical needs."[2]

Another passage in the same work (p. 45) contains a description of Comte's general view of method: "Universal method consists of three elements: deduction, induction, and construction, whose order is here represented in accordance with the increase in importance and difficulty. Deduction is immediately possible wherever the speculations are so simple that their principles can be spontaneously grasped. The application of induction is determined by the degrees of complexity exhibited by phenomena; it prevails wherever the establishment of starting points is more difficult and of more value than the development of consequences."

We shall have occasion to see later to what extent this view agrees with the methods "really employed" in science.

[1] 2nd ed., Paris, 1900.   [2] Op. cit., p. 27.

32. J. F. W. Herschel's Discourse on Natural Philosophy

J. F. W. Herschel takes up the problem of scientific method [1] in a spirit quite similar to that of Comte, although he does not extend his positivistic criticism to the metaphysical hypotheses underlying scientific constructions.

The author, the son of the prince of astronomers and himself famous in astronomy, especially for his revision of the whole work of his father, possessed a wide knowledge of the most various scientific theories and the vivid feeling of the scholar who actually collaborated in the discovery of truth. It is exactly the history of certain famous examples (such as Fresnel's optics) that made him realize "the large part reason has to perform in our examination of nature" (no. 23). At the same time he proclaims experience "as the great and ultimate source of our knowledge of nature and its laws," and he insists upon the fact that by experience he means "not the experience of one man only or of one generation, but the accumulated experience of all mankind in all ages, registered in books and recorded by tradition" (no. 67).

But how does Herschel explain the relationship existing between the rational and empirical elements of knowledge?

He distinguishes (no. 14) an abstract science whose objects are "first those primary existencies and relations which we cannot even conceive not to be, such as space, time, number, order, etc.; and secondly those artificial

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[1] *A Preliminary Discourse on the Study of Natural Philosophy.*

forms or symbols which thought has the power of creating for itself at pleasure, and substituting as representatives, by the aid of memory, for combinations of those primary objects and of its own conceptions . . . , such as language, notation, and that higher kind of logic which teaches us to use our reason in the most advantageous manner for the discovery of truth; which points out the criterion by which we may be sure we have attained it, and which by detecting the source of error, and exposing the haunts where fallacies are apt to lurk at once warns us of their danger and shows us how to avoid them."

This higher kind of *logic* which he proposes to designate as *rational* (in contradistinction to *verbal logic*) does not permit us to predetermine nature, because the study of nature is dominated by the idea of *cause* which does not enter into abstract science (no. 66). Also "the obscurity which hangs about the only act of direct causation of which we have an immediate consciousness will suffice to show how little prospect there is that in our investigation of nature we shall ever arrive at the knowledge of ultimate causes." We have therefore to "limit our views to the discovery of *laws* and to the analysis of complex phenomena into simple ones, which, appearing to us as incapable of further analysis, we must consent to regard as causes" (no. 78).

A law of nature will then present itself under two aspects, namely as:

1) a general proposition announcing in abstract terms a whole group of particular facts relating to the behavior of natural agents in proposed circumstances;

2) a proposition announcing that a whole class of individuals agreeing in one character agree also in another.

The study of these laws leads our author to a discussion of induction, where he starts with its first grade, which is the discovery of proximate causes and the establishment and the verification of the laws of the lowest grade of generality (no. 137 ff.). To this end he analyzes the causal relation (no. 145). He finds here an invariable connection between an antecedent (cause) and a consequent (effect). This means that the absence of the cause involves the negation of the effect, unless some other cause is capable of producing the same effect, and that the increase or diminution of the cause involves a corresponding variation of the effect and also a proportional variation in all cases of direct and unimpeded action.

This examination establishes the rules to be followed in philosophizing (no. 146), namely the methods of inductive research. These are for our author the method of agreement (the cause inferred as a common antecedent of similar effects), the method of disagreement, which tells us to take into account contrary as well as favorable facts in the discovery of causes (no. 158), and the method of residues (no. 158). All these methods are illustrated by means of numerous scientific examples.

The methods of research proposed by Herschel do not, however, lay claim to demonstrative rigor. We are rather admonished not to be too exacting in our criticism of inductions and to accept them provisionally pending their eventual *verification* (no. 170). The value even of a complicated theory is ultimately to be measured by its ability to *predict facts before trial* (no. 215).

It is this criterion which makes it possible for us to reach the highest inductions of science by means of successive generalizations.

"The safest course," he says in no. 217, "is to rise
by inductions carried on among laws, as among facts,
from law to law, perceiving as we go on, how laws which
we have looked upon as unconnected become particular
cases either one of the other, or all of one still more
general and at length blend altogether in the point of
view from which we have to regard them."

33. THE "FUNDAMENTAL IDEAS" IN THE LOGIC OF WHEWELL

Such an examination of the inductive procedure of
science gives rise to certain philosophical problems to
which, we must admit, Herschel did not pay sufficient
attention. How can reason assert itself over against
experience? What is the character of the basis un-
derlying abstract science by means of which we in-
terpret experimental data? [1] What is the nature of the
idea of cause and effect which we assume as a criterion
of the inductive method? All these questions indeed have
not been critically investigated by Herschel. The man-
ner in which he regards the latter idea greatly suggests
the speculations of the preceding critical philosophers,
and especially those of David Hume, from whom he bor-
rowed the positivistic conception of the causal relation
as "an invariable succession." But at the same time he
retains something of the dogmatic tradition in his con-
ception of causality. In any case the combination of two

[1] These questions are at least not examined in the above-mentioned
discourse. Later, when Herschel comes to criticize Whewell's doctrine
in an anonymous article in the *Quarterly Review* of June, 1841, which
was republished in his essays, he expounds views of a more decided
empiricistic nature, in the sense of John Stuart Mill.

apparently contradictory tendencies in Herschel's scientific conception had to call forth a wider discussion in this part of logic. The two opposed points of view are expressed and represented by William Whewell and John Stuart Mill.

Whewell, who was inspired by the Kantian philosophy, devoted to inductive logic several fundamental works:

History of Inductive Sciences . . . (1837).

The Philosophy of Scientific Ideas (London, 1840; the second part was republished later under the title of *Organum renovatum*).

History of Scientific Ideas (London, 1858).

Whewell distinguishes two inseparable and irreducible elements in all knowledge: facts and ideas. The object of science is to connect phenomena by means of ideas. The *fundamental ideas,* however, are not of an empirical origin, although they are involved as "forms" in sensations. We have here a metaphysical element derived from an analysis of knowledge and of the history of science. The ideas are comprehensive forms of thought which we apply to phenomena; the concepts are special modifications of such ideas; the circle, for instance, is a modification of the idea of space. The function of explanation of concepts and of observation of facts is to prepare the intellectual and sense materials of science; induction puts them into actual operation. Induction establishes a true bond between facts by means of exact and appropriate concepts. It is thus neither a pure sum of facts nor their connecting idea but the act by which mind introduces a unifying intellectual element into scattered and different facts.

The analysis of induction rests upon a penetrating criti-

cism of the concept of causality. In regard to this point we shall especially follow the exposition given in the last of the above cited works (Vol. I, book III). Our author insists that the idea of cause is not derived from experience, because it involves a universal assertion: every event must have a cause. A cause is something more than an antecedent or an occasion, for it is conceived as a power which has a real operation (p. 176). After examining the opinions of various philosophers and especially the discussion called forth by the Humean doctrine, Whewell adopts and clarifies the Kantian conception (p. 180). The idea of cause is inseparable from the conditions of every possible experience. Like the fundamental ideas of space and time, this idea is one of the *active* powers of our mind (p. 183). In order to illustrate its significance he states the axioms of causality (necessity of cause, proportionality of the causes to the effects, at least in the case of additive elements, equality of action to reaction), referring to the premises of the Newtonian mechanics. He then goes on to examine in a detailed manner the physical theories which can be treated as applications or extensions of this mechanics. Each of these theories is made to rest upon a fundamental idea which is brought into relation with the primary suppositions of the given theory; this is to be completed and developed by means of appropriate explicit hypotheses. Optics, for instance, depends upon the fundamental idea of a medium in which luminous undulations are propagated (p. 293).

In his examination of the presuppositions of Newton's mechanics Whewell shows himself to be a faithful interpreter of Kantianism. But it is exactly because of this

that his criticism brings to light the shortcomings of
Kant's philosophy. This is especially seen in those places
where he is carried away by his zeal to the extent of
justifying action at a distance. He maintains here that
there is no necessity for admitting continuity of causal
action in space, as we do in the case of time (p. 277). A
rigorous advocate of the a priori ought not to have yielded
to the all too facile explanations of the Newtonians in
regard to a point so fundamental for the requirements
of knowledge, even at the risk of finding himself in the
Leibnizian camp.

But this accommodating attitude on the part of our
philosopher is in a certain sense necessitated by the very
Kantian doctrine of the a priori. It is assumed, on the
one hand, that there exists a science ready made (like
the Newtonian physics), and, on the other hand, that
the possibility of science depends upon the acceptance
of certain necessary principles which are the products
of the order introduced by the activity of mind into
the data of experience. The implication is thus that the
principles of science which are regarded as definitive
acquisitions must simply agree with the conditions of
experience and receive from them an absolute and uni-
versal justification. For the absence of this agreement
would spell destruction to science.

This shows clearly the danger involved in the regres-
sive method of the Kantian critique. The contingent—
and necessarily also empirical and approximate—suc-
cess which an historically constituted scientific doctrine
(be it that of Euclid or that of Galilei-Newton) may
have achieved *in fact*, is here transformed into a justi-
fication *by right*. Understood in this sense, the a priori

ceases perhaps to be the inconvenient pretender that tries to impose the authority of reason in collaboration with and often in opposition to the development of experience. It will turn out to be instead a useless confirmer of what has already received its justification from other sources, from the success of experimental proofs. And it may have to share the fate of the servant who is foolish enough to be so complaisant as to offend from the very beginning the new master who is to determine his fitness.

Seen in the light of history and of the examination of scientific doctrines, the development imparted by Whewell to the Kantian philosophy had naturally to call forth an empiricistic reaction against every form of rationalism. This finds its expression in the critical views of Herschel and especially in the logic of John Stuart Mill.

34. The Inductive Logic of John Stuart Mill

John Stuart Mill's work, *A System of Logic Ratiocinative and Inductive,* which was published first in 1843 and republished afterwards in successive editions and translations, owes its great success to the admirable lucidity of thought and exposition and also to the simplicity of the decidedly empiricistic point of view of the author.

The methodology of Herschel is here given a more schematic form with the same purpose of constituting a *logic of truth* (objective). The *logic of consistency*—dealing with the agreement of thought with itself—is assigned, on the other hand, only an auxiliary value. It is of course true that our author does not leave out of

account questions referring to this aspect, for instance, the expression of thought by means of language, the nature of definition, etc. Here one can perhaps detect the happy influence exercised upon Mill by Gergonne's lectures on logic, which he attended, as he himself tells us in his autobiography.

Logic of truth means for Mill not only logic of discovery but also logic of *proof*. Its basis is to be found exclusively in experience where no a priori principles are admitted. The battle against such principles is waged vigorously by means of a thorough examination of the so-called deductive sciences. In the first place, there is properly speaking no deduction proceeding from the general to the particular in opposition to induction rising from the particular to the general. For all ratiocination proceeds in reality by analogy from particulars to particulars. Indeed (as Sextus Empiricus had already pointed out against the Aristotelian logic) the major premiss of the syllogism:

> All men are mortal,
> Socrates is a man,
> Therefore Socrates is mortal

could not be asserted as true unless we were first convinced of the truth of the conclusion. Far from forming the elementary type of reasoning, the syllogism presents only its scheme or touchstone (Book II, Ch. II, III).

In the second place, mathematical sciences do not rest upon necessary truths but only upon hypotheses and certain axioms which are generalizations from experience (Ch. V, VI). Hypotheses are for our author deformations of real objects obtained by the omission or exaggeration of certain aspects (for instance, lines without

width). Axioms, on the other hand (for instance, "two straight lines cannot enclose a space") are truths arrived at inductively on the basis of experience and involving the notion of limit.

Induction rests upon the principle of *uniformity of nature,* which is regarded as a most general fact gained by means of a primitive induction *by simple enumeration,* and which is capable of yielding the criteria of the particular inductions upon which we build science. To be more precise, uniformity of nature resolves itself into a tissue of partial regularities of phenomena whose threads are the *constant and unconditioned sequences* which we call relations of *cause* and *effect* (Book III, Ch. III, V).

The mental analysis of complex phenomena into their elements forms the first step of inductive inquiry (Ch. VII) and leads the author to define the methods of Herschel more accurately and to formulate their general canons (Ch. VIII). These methods are reduced to four:

1) The method of agreement,
2) The method of Disagreement,
3) The method of residues,
4) The method of concomitant variations.

In order to establish these methods the author utilizes the scheme of phenomena which are analyzed, as we have pointed out, into their elements ABC, ABC . . . , *abc, ade* . . . For instance, if ABC is followed by *abc* and ADE by *ade,* we may conclude that A is the cause of *a* (method of agreement), etc. The examples are drawn by preference from elementary scientific theories (Ch. IX). Particularly he borrows from Herschel the example

of Wells' explanation of dew, which has since been so often reproduced in numerous logical treatises that it has become more famous in this field than in that of science itself.

After exhibiting the various inductive methods, John Stuart Mill takes up the question of the plurality of causes and of the intermixture of effects (Ch. X). It is here that he finds the explanation of the deductive method. The process of deduction consists for him of three stages:

1) First of a direct induction for the ascertainment of the laws of the separate causes;

2) Then of ratiocination from the simple laws to the complex cases;

3) And finally of verification by specific experience (Ch. XI).

In Ch. XIV of the same book he clarifies this process by comparing it with certain physico-mathematical theories, in which the application of the deductive method rests upon hypotheses or upon hypothetical representations of phenomena (for instance, Fresnel's undulatory optics). Here the author repeats the old Aristotelian argument to the effect that true consequences can be deduced from false premises (§ 4) so that the verification of the conclusions does not offer in general a justification of the hypotheses. And as far as representative hypotheses are concerned he is guided by the positivistic criterion of Auguste Comte according to which they cannot lay claims to true scientific explanations.

35. DEDUCTION AND INDUCTION COMBINED BY STANLEY JEVONS IN THE PROCESS OF INFERENCE

Postponing the examination of this point, we may say that Mill's attitude towards deduction betrays a mind insufficiently trained in mathematics.[1]

A thorough analysis of the method of this science would have taught him that deduction does not necessarily proceed from the general to the particular and that the traditional distinction between deduction and induction has therefore no basis. But above all the sense of mathematical exactness would have prevented him from regarding as methods of rigorous proof those comparisons and analogous generalizations from experience which are in reality mental operations quite different from reasoning and which were regarded by Herschel as methods of research only.

Had he realized that all operations of this kind are always and only approximate and probable, he would have come to the conclusion that inductive results are nothing but suppositions whose probability can increase and come near to certainty only thanks to more and more extended verifications to which we are led by deductive reasoning. In this way he would have appreciated the true value of this method of reasoning, which had become already for Kepler and Galilei—owing to their rejection of the narrow Aristotelian criterion of the impossibility of regarding consequences as proofs of

[1] What our author lacks in order to attain to the rigor of a logic based on mathematics can be seen, for instance, from his analysis of the demonstration of the fifth proposition of Euclid (the equality of the base angles of an isosceles triangle). Cf. Book II, ch. IV, § 4.

premises—an integral part of the inductive process by which knowledge is acquired. The criticism of the great scientific theories would have shown him the function of negative verifications, which serve to correct partial errors found in premises and to replace these by more general conclusions.

Ultimately Mill's conception of science—in which he follows Telesio and Bacon—as a progress from the particular to the general is rendered more significant by such considerations. It offers a rigorous affirmation of the inductive character of knowledge in opposition to the old deductive ideal. A problem is thereby raised which could no longer be solved in the old Aristotelian sense, to which Newton still seems to have clung (in the passage of the *Optics* quoted in § 31). What part do deduction and induction play respectively in science?

The true combination of these two methods, namely their explanation as subordinate parts of scientific procedure, has been offered by a countryman of Mill. W. Stanley Jevons was like Mill greatly interested in the social and economic sciences, into which he introduced (after Cournot) the mathematical method. In his description of inference [1] Jevons distinguishes four stages:

[1] *The Principles of Science*. A treatise on Logic and Scientific Method, 1873. In the preface (p. XXII of the 2nd ed.) the author criticizes Mill's psychological and philosophical treatment of logic. By comparing the explanations given by the two authors of the principles of logic (of contradiction and excluded middle) we can form a good idea of the difference in their views. For Mill these principles simply express the psychological experience that belief and disbelief are mutually exclusive states of mind and that we must therefore believe or not believe in all cases which are not meaningless (Book II, ch. VIII, § 4). For Jevons, on the other hand, logical principles are logical conditions which have to be satisfied by the objects of our reasoning processes, as we have already pointed out in § 30.

preliminary observation (which can be replaced by experience arrived at by previous reasoning), hypothesis, deduction, and verification. It is noteworthy that experimental reasoning is described in the same way by Claude Bernard in his famous *Introduction à la médecine expérimentale.*

36. THE ARBITRARINESS OF THE CAUSAL ANALYSIS

As Vailati justly remarks,[1] the works of Jevons and Bernard bring to light a tendency to rehabilitate in a certain sense the *constructive* and *anticipatory activities* of the human mind against the purely receptive and, so to speak, recording and classificatory activities. In the past an all too important and especially too exclusive part had been assigned to the latter in the mental processes aiming at the discovery of truth. This activistic conception is becoming more and more prevalent owing to the influence of certain general factors. We shall examine here briefly the directions assumed by it.

A consistent empiricism must necessarily break up the concept of uniformity of nature into separate fragments and hence decompose the postulate of causality which Mill regarded as the basis of induction.

As we have pointed out, Mill analyzes the concept of the uniformity of nature into a multitude of partial regularities formed out of the constant and unconditioned sequences of phenomena which are regarded as causal relations. It is this idea that enables us to represent phenomena as combinations of simple elements in accord-

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[1] *Scritti*, p. 283. See also by the same author: "Il metodo deduttivo come strumento di ricerca" (1898) in *Scritti*, p. 118.

ance with the scheme employed by the author in his treatise. But this decomposition of phenomena into their elements is relative and arbitrary. The circumstances accompanying a given phenomenon can hardly be defined by means of a finite number of data, since they are in reality infinite. Among these there certainly can be found permanent conditions of nature (or of the terrestrial environment) which deserve the title cause—in the case of numerous physical phenomena—just as much as those invariable antecedents upon which we usually bestow this name. In the most developed physical theories, for instance, in mechanics, we indeed do not speak of cause and effect but rather of the *interdependence* of the distinctive peculiarities of phenomena. This point of view has been developed by Ernst Mach since 1872.[1] Instead of saying in vague terms that a certain phenomenon or phenomenal character is the cause of another, mathematical language translates their relation by calling one the *function* of the other. A phenomenon in this way is generally determined by a certain number of functions of several variables each of which can appear as a cause.

It follows from this that causes and effects express concatenations of phenomena in the direction which is most *important* for us to study. The idea of the necessity of the cause is a psychological fact which Mach tries to explain after the Humean fashion by connecting it with voluntary movements.[2] "What we call cause and effect," says Mach in his treatise: *Oekonomische Natur der*

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[1] *Die Geschicte und die Wurzel des Satzes der Erhaltung der Arbeit* (Prag, 1872).

[2] *Die Mechanik in ihrer Entwickelung* (1st ed., Leipzig, 1833). Ch. IV, V. Translated into English by T. J. MacCormack as *The Science of Mechanics* (Chicago, 1907).

Physischen Forschung,[1] "is nothing but the chief characteristics of an experimental datum. It is owing to the selection made of these qualities by our mind that some of them appear as more important than others." He adds that the question "why?" can go beyond our aims and be raised in cases where it becomes meaningless.

A more precise exposition of Mach's ideas (a substitution of the causal relation by a functional dependence, capable of determining phenomena in a univocal manner) is to be found in I. Petzoldt's article "Das Gesetz der Eindeutigkeit."[2] But these ideas have been especially espoused by the representatives of the statistical and economic sciences. The application of the mathematical method has enabled these scientists to arrive at a more exact and comprehensive view of the interconnectedness of social phenomena and thus to combine the partial explanations of the preceding economists and sociologists into a higher synthesis. I refer particularly to Karl Pearson and Vilfredo Pareto.

Pearson has been led from biometrical and statistical researches to general reflections on the theory of science, which find their expression in his important philosophical work *The Grammar of Science*[3] (1892). The book seems to be influenced in its main tendencies by the critical views of Clifford and especially by the ideas of Mach (who is often quoted).

[1] *Populaer-wissenschaftliche Vorlesungen*, 3rd ed., p. 277. Translated into English by T. J. MacCormack as *Popular Scientific Lectures* (Chicago, 1895).

[2] *Vierteljahrsschrift fuer Phil.*, XIX, 1895. Cf. *Einfuehrung in die Philosophie der reinen Erfahrung*, by the same author (Leipzig, 1900) (a development of the philosophy of Avenarius).

[3] 3rd ed., 1911.

As far as the notion of cause is concerned Pearson denounces it as a "fetish" lying "amidst the inscrutable arcana of science." "Is this category," he asks in the preface to the third edition of his work, "anything but a conceptual limit to experience and without any basis in perception beyond a statistical approximation?" As the work proceeds the ideas of our author are developed in a characteristic manner. "The universe," he says in Ch. V, § 5, "is made up of innumerable entities, each probably individual, each probably non-permanent. All man can achieve is to classify by measurement or observation of their characteristics these entities into classes of like individuals. Within these classes variations can be noted, and the fundamental problem of science is to discover how the variation in one class is connected with or contingent upon the variation in a second class." And a little further (§ 8): "That the universe is a sum of phenomena, some of which are more, others less closely contingent is a conception wider than that of causality which we may at the present time draw from our widening experience."

Closely connected with this view is the thought expressed by Henri Poincaré in Ch. XI of *La valeur de la science* [1] (1905), in which he examines the question of contingency and determinism, concluding that causality can be reduced to a "classification of sequences."

In *Science et méthode* the consideration of the causal connection as a functional relation between two elements is given by the author an interesting application in connection with the definition of "chance" (Ch. IV).

[1] Translated into English by G. B. Halsted as *The Value of Science* (New York, 1907).

It is evident that our author's conception of causality makes it impossible for us to retain the axiom of Herschel, Whewell, and Mill, which says that "causes are proportional to effects." Indeed if the element y depends on x in accordance with a certain function $y = f(x)$, there is no a priori reason for reducing f to a linear function of the type $ax + b$. If, however, we admit that f is a continuous function and that it has a derivative that is finite and different from zero, we can have: $f(x + h) = f(x) = hf'(x)$ as a first approximation for a sufficiently small h, and *hence the small variation of the cause (x) can be said in general to be proportional to the small variations of the effect (y).*

It may happen of course that f' becomes zero or infinite. In the first case no sensible effect corresponds to the small variations in the cause; in the second case great effects correspond to small variations in the cause. It is here that the phenomenon of "chance" intervenes. A good illustration of this is offered by the game of roulette. The fact that the wheel stops at the white or the black—which is precisely a variation in the effect that matters to the player—depends on the minimal and imperceptible variations in the impact that has set the disk in motion. The conjunction of different causal series—the brick which falls upon the head of the passer-by (namely the definition of chance offered by Cournot)—is also reduced by Poincaré to this explanation.

We may also mention another aspect of the conception of the arbitrariness involved in the choice of causes.

According to Vailati (1903), the interest which is instrumental in the selection of a cause out of the determining

conditions brings to light the influence exercized by feeling in the historical sciences.[1]

We too (although in a spirit somewhat different from that of just mentioned writers) have pointed out the arbitrariness attached to the determination of causes and have shown its significance from the point of view of one who is interested in reproducing experimentally a given phenomenon. The negative aspect of this analysis throws light particularly upon the juridical problem of responsibility, where we regard as causes those human actions which are instrumental in the commission of crime and which it is especially important to prevent.[2]

37. THE VALUE OF CONCEPTS: THE ECONOMIC DOCTRINE OF SCIENCE

We have seen how impossible it is to disengage causal series from the tangled skein of interdependent natural phenomena. It is similarly impossible to verify separately the hypotheses or principles of a theory when they are submitted to an experimental verification. In the absence of verification we are not in a position therefore to determine which among the hypotheses introduced has been disproved by experience. This is why Mach was led to express himself in regard to Mill's criteria of induction in the following way: [3] "I can imagine the feeling of every modern student of nature who, for instance, keeps constantly in mind Mill's methods of experimental

[1] *Scritti,* p. 463.

[2] *Problems of Science,* ch. III, B.

[3] *Analyse der Empfindungen* (1886), ch. V. Translated into English by C. M. Williams as *The Analysis of Sensations* (Chicago and London, 1914).

research. The attempt to apply them would not lead him beyond the most provisional cases."

The recognition of the interconnectedness of the factors involved in the determination of phenomena implies that we have to modify not only the criteria of Mill but also the scheme of inference of Jevons and Bernard. Instead of describing the second stage of this scheme as "hypotheses derived from observations or preliminary experiments" we shall have to regard it as consisting "of the assumption of a concept or system of concepts which serve hypothetically to represent the observed data." [1]

We thus come back to the Democritean principle:

$$\text{ἔννοια κριτήριον ζητήσεος.}$$

The question of causality in this way is seen to be connected with a more general problem: what value have the concepts with which we try to represent reality?

This historical problem had been solved by Berkeley in a nominalistic sense before Hume was led to his famous analysis of causality. It is the position of Berkeley and the ancient nominalists that had to be resumed by the thinkers who wanted to give a deeper meaning to scientific explanation. Along with Mach we have to mention in this connection I. B. Stallo and his *Concepts and Theories of Modern Physics* (1882). Mach himself sums up the results of this work in Ch. VIII of *Erkenntnis und Irrtum*, pointing out their similarity to his own views.

It is true, both Mach and Stallo have the same end in view, the destruction of mechanistic metaphysics, and both resort to an extensive examination of scientific doctrines. The coincidence in their views is, however, all

[1] Enriques, *Problems of Science*, ch. II.

the more instructive as they develop their theme in quite different manners. Mach is inspired by a biological conception of science, while Stallo starts with a critical examination of contemporary philosophers (Mansel, Whately, Hamilton, John Stuart Mill, etc.).

Common to the two philosophers is, as we have pointed out, the nominalistic thesis: there are nothing but phenomena in reality. But this conception does not lead them to the extreme position of Roscellin according to which concepts are only "flatus vocis." Concepts are for them mental constructions whose significance and value has to be established. Thought, according to Stallo,[1] is the establishment and recognition of relations between phenomena. Foremost among these relations—the foundation of all others—are identity and difference.[2] It is from the latter that we derive exclusion and inclusion as well as cause and effect. Objects are *perceived* as different, they are *conceived* as identical by the attention of the mind to their point or points of agreement.

This process of abstraction leads us to classify the objects of our knowledge in accordance with a scale of concepts which ascends from the "infimæ species" to the highest class or "summum genus." Conceptual operations are connected with certain traditional errors which metaphysicians derived from common sense, namely:

1) Every concept is a counterpart of objective reality, and, consequently, there are as many things as there are concepts.

[1] Op. cit., ch. IX.
[2] This view is extensively developed in the whole English psychology (Bain, Lewes, Spencer, etc.).

2) The more general or extensive concepts are prior to the less general.

3) The order of the genesis of concepts is identical with the order of the genesis of things.

4) Things exist independently of their relations.

To these errors Stallo opposes views derived from his own analysis:

1) Thought deals not with things as they are but with our mental representations of them, that is, with concepts.

2) Objects are known only through their relations to other objects.

3) An operation of thought never involves the entire complex of the known or knowable properties of a thing but only such of them as belong to a definite class of relations.

We have already said that Mach arrived at similar conclusions. He returns again and again in his principal works to a doctrine of concepts inspired by a biological and economic point of view.[1]

A concept, he says in *Erkenntnis und Irrtum,* does not correspond to a "representative, intuitive, concrete, actual content that exhausts completely its meaning." We have not, however, to regard it as "flatus vocis." In reality it expresses a psychological *construction* which, unlike a sense representation, is not instantaneous; it often takes quite a long time. Human beings form concepts in the same way as animals, being guided by a biological interest

[1] Cf. *Die Mechanik,* ch. IV, V; *Analyse der Empfindungen* (4th ed., p. 249, 55); *Prinzipien der Waermelehre* (2nd ed., pp. 177, 80); *Erkenntnis und Irrtum* (2nd ed., pp. 126, 143); *Oekonomische Natur der physichen Forschung* (*Populaer-wiss. Vorlesungen,* 3rd ed., 1903, p. 277).

in the retention of certain characteristics of entire classes of phenomena. They are, however, helped by language and by relations to their fellow beings. Words are actually labels enabling us to grasp concepts in a sensible form even when the representations become insufficient. Most human beings, by whom the use of words is determined, pay attention only to "a small number of *biologically important* reactions, and it is through this that the employment of words becomes stable." Finally if the common characters of objects represented by concepts are not actually intuited "with the feeling that they are *reproducible,* then *potential intuition* has to take the place of actual intuition."

Owing to these circumstances concepts become very precise and useful tools for the purpose of representing and symbolizing in thought large classes of facts. It is here that Mach's splendid economic conception makes itself felt.

Thought is a function of the living organism. The tendency to preserve life and to save vital energy so characteristic of all physiological processes comes to light also in thought. But it is precisely because the conceptual reproduction of phenomena is not *global* but partial embracing only the most *important* aspects which determine the choice of causes that the possession of general concepts enables us to leave out of account all the properties that are irrelevant from the point of view of a given purpose.

On the other hand, science is a form of activity, to be sure, not of separate individuals but of human society; it has thus to obey the same economic laws that govern all forms of work. Biology and economics conjointly

throw light on the nature of scientific concepts. "The aim of every science," says Mach in his *Mechanik*, "is to replace and *save* experiences through the reproduction and anticipation of facts in thought." The economic function which permeates all science already manifests itself in general demonstrations. It becomes, however, more evident when we examine particularly the developments of science.

These simple views are highly suggestive. Any one trying to compare the developments of science with those of industry will realize how far the analogy can be pushed. The various meanings which can be given to economic law will then become obvious. A justification will be found in this way not only for the construction of the most general concepts and hence for that of the most comprehensive branches of knowledge but also for the efforts directed in every branch towards the purity of methods. The latter are analogous in many respects to those organizations of industry where it is found more economical to retain existing plants with a smaller output than to erect new plants with a larger production.

38. Phenomenalism and the Definition of Reality

The biological conception of knowledge exercized its influence upon a philosopher whose name is generally associated with that of Mach in the records of empirocriticism. We have in mind Richard Avenarius, the founder of the great movement represented by the *Vierteljahrsschrift fuer wissenschaftliche Philosophie*. His theory of knowledge is to be found in the two volumes of

his *Kritik der reinen Erfahrung* (Leipzig, 1888-89). In the introduction he refers to the works of Mach which had a great influence on him. Mach in his turn proclaimed his indebtedness to Avenarius whose publications helped to gain the favor of the public for his own ideas. We must, however, admit that the form in which Avenarius expounds his ideas is hardly a happy one, as it is encumbered with a difficult and heavy terminology.

Like Mach's analysis of sensations, the criticism of pure experience aims to give expression to a radical *phenomenalism*. When we decompose knowledge into its elements, which are the immediate data of reality, we arrive not at "things in themselves," which are for us entirely meaningless[1] but only at *sensations* and the relations by means of which they are associated. The definition of reality thus reduces itself to Berkeley's formula: *esse = percipi*.

We must, however, say that the assertion of the real existence of an object implies not only the possibility of certain sensations but also the fact that we can produce and reproduce them at will. This conception has been reached by J. Pikler in his work *The Psychology of the Belief in Objective Existence* (London, 1890), especially through a criticism of our judgments referring to existence in space and time. According to Pikler these judgments imply that we are able to represent, by means of the voluntary movements of our eyes, not only the parts of space that are given in our experience at a certain

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[1] We try to show in ch. I of our *Problems of Science* that "the thing in itself" as well as the "absolute" are concepts defined fallaciously by means of a transcendent process implying infinite degrees of knowledge or acts of thought.

moment but also other parts lying to the right, left, above or below.

In Ch. II of the *Problems of Science* (1906) we have been led independently to the same view,[1] understood in a wider sense, as a "definition of reality." Our starting point was a desire to find a criterion for distinguishing the world of reality from that of dreams and sense illusions. For us the immediate data of reality are thus not pure sensations but rather relations between sensations and volitions which condition our expectations and which express their elementary invariants. The assertion of the existence of any object always implies the recognition of such an invariant which is relative to a system of sequences between certain acts or voluntary movements and the sensations produced by them.

This view (which solves in a positive sense the difficulty raised by Berkeley's criticism) enables us, as we shall see, to synthetize the notion of "brute fact" and that of "scientific fact" into the highest interpretation of the term. The development of the *radical empiricism* represented by William James and Henri Bergson leads, in opposition to this conception, to a sharp separation between immediate reality and the world constructed by science, which they regard as abstract and artificial. A more thorough analysis of experience indeed reveals that sensations, far from having a meaning "by themselves," can be defined only in relation to a state of consciousness as its variations or differentials. They will be found to be contained in consciousness as expectations conditioned by voluntary acts, and, if we may say so, in the same way

[1] This is why the name of Pikler is cited only in the second edition of 1908.

as the characteristic behavior of a mechanical system is contained, with reference to its virtual deplacements, in its equilibrium.[1] A state of consciousness cannot be grasped synthetically as an instantaneous view taken by mind of itself. It evaporates in the act of reflection; the very idea of such a state is nothing but a limit of a series of mental constructions. From whatever angle we may approach this we must admit that it is not given to us to reach anything which is truly "primary" and which does not imply a possible variation and hence a relation between sensations and associated volitions. This variation thus proves to be the ultimate limit of the immediate reality which thought can reveal to us.

39.   THE SIGNIFICANCE OF SCIENTIFIC HYPOTHESES

Let us come back to Mach and to the philosophers of his school in order to find out what value and significance scientific explanations possess for them. The answer given by them to such a question is essentially inspired by the positivistic spirit of Auguste Comte. The content of knowledge is made up solely of experimental facts. On its highest level science can embrace many facts and help us to predict them in an economical manner. It can, however, never transcend the limits of possible experience. "The first manifestations of knowledge," Mach says in his popular scientific lectures, "are born of the econ-

[1] Only in the case of fugitive impressions is it possible to speak of pure sensations. In every sensation there will always be found an active reaction of the sentient, a voluntary effort of attention which is indissolubly fused in consciousness with the sense element and which is apprehended as a "cognition."

omy of self-preservation. This circumstance determines the whole mysterious power of science."

What Mach condemns under the name of "mechanical mythology" is simply the use of hypothesis owing to which the laws of motion are carried beyond visible phenomena, and physics is thus regarded as an extension of mechanics. No one was more influential than our author in the realization of this positivistic program. Suffice it to say that it was he who inspired the battle which Ostwald waged in favor of energetics against atomism, although Ostwald had to retreat from his position as a result of the admirable discoveries of Perrin. Besides, the long studies which he devoted to the treatment of an optics dispensing with the ether [1] bear testimony to the interest he took in such questions. Some passages of his works show, however, that in a certain sense he recognized the value of hypotheses as instruments of research. Thus we read in Ch. IV of *Erkenntnis und Irrtum*: "The Ionian and Pythagorean animistic and demonological mythology of nature gradually yielded place to a mythology of substances and forces, then to a mechanistic and automatic mythology, and finally to a dynamic mythology." He mentions as examples the atoms of Democritus and Dalton, the luminous particles of Newton, and the electrons of modern physics. He ends by saying: "Innumerable flowers of the imagination must succumb to the facts of an inexorable criticism before one of these flowers has a chance to develop further and to reach a durable state. Before we can *understand* nature

[1] See his posthumous work, *Die Prinzipien der physikalischen Optik,* with a preface written by the author in 1913, published in Leipzig in 1921. Translated into English by A. F. A. Young and J. S. Anderson as *The Principles of Physical Optics* (New York, 1925).

we must *grasp* it in the imagination in order to give concepts an evident intuitive content." The imagination is therefore the more vivid the *further* the problem to be solved is removed from a biological interest.

He comes back to this subject in the chapter of the same book devoted to the question of hypotheses. He criticizes here the requirements which scientific research has to satisfy for Mill and reaches the following conclusion: "A pure conceptual representation can be applied to the closed parts of science where there is no place for hypotheses, as these have a function only in science in the making. But the use of images is quite *useful*."

As to the fully developed science it consists entirely of economical *descriptions* of phenomena, as was maintained by Kirchoff who assigned to mechanics the task of *describing* the motions of natural bodies in the simplest terms possible and with the smallest number of hypotheses. For Mach (in the chapter of *Erkenntnis und Irrtum* devoted to the "laws of nature") the result of science can be defined still better, as a "limitation of expectations," thus indicating its biological value: "According to their origin the laws of nature are limitations which we prescribe to our expectations under the guidance of experience." "Science can be considered as a *collection of instruments* enabling us to complete by means of thought those facts which are given to us only in part, or to limit our eventual expectations as far as it is possible."

Along with Mach and notably under his influence other thinkers combated the metaphysical hypotheses which cannot be translated into possible experience, and especially the mechanistic metaphysics. Thus Stallo [1] shows

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[1] Cf. *The Concepts and Theories of Modern Physics.*

with great vigor the contradictions involved in the hypothesis of absolutely hard and inelastic atoms as well as in that of their absolute inertia, etc. He ends his criticism in Ch. VII by setting forth the conditions which can make scientific hypotheses valid. The task of .hypotheses is according to him to reduce the phenomena to be explained to others which are familiar and already established and with which they are to be brought into relation.

Pearson too maintains and develops similar views. Matter and force are fetishes. And is not the scientist of to-day (he asks in the new edition of the *Grammar of Science* of 1911) confronted with the danger of treating the electron like the old and immutable atom, forgetting that it is only a "construct of his own imagination" which will have to give way to a wider concept when its splendor will have been worn off? "Science," he repeats in Ch. VI, "takes the world of perceptions as it finds it, trying to describe it briefly. It does not assert the perceptual reality of its own shorthand."

Parallel to the critical movement which we have just described an evolution has been taking place in the thought of the mechanistic physicists themselves. Although Stallo refers the mechanistic views of Clerk Maxwell to the class represented by Descartes, Huyghens, and Leibniz, we have to admit that Maxwell's conception tends to emancipate itself from the ancient materialistic metaphysics. "When a physical phenomenon can be described completely as a change in the configuration or the motion of a material system," he says in an article in *Nature*,[1] "we say that the dynamical explanation of this phenom-

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[1] "On the Dynamical Evidence of the Molecular Constitution of Bodies," the 4th and 11th of March, 1875.

enon is complete. We cannot conceive the necessity or possibility of a further explanation," since the ideas of configuration, mass, force are so elementary that they cannot be explained by any other notion.

In agreement with such a view Maxwell made numerous attempts at explaining various orders of phenomena in a mechanistic fashion, especially in the field of electricity. He offers, however, his explanations not as a final solution of the problem, which would go beyond the positivistic point of view, but as provisional theories having the value of predictions, even if they mutually disagree. In this way Maxwell, like Lord Kelvin, is led to look in physical theories for *mechanical models* of reality.

This aspect of Maxwell's construction has been brought clearly to light by Poincaré.[1] Our author points out that the English physicist generally tries to show "the possibility of a mechanical explanation," without claiming to reach reality. He then proves that it is possible to assign to a series of phenomena an infinite number of explanations if it is possible to give it a mechanical explanation.

By regarding scientific theories in general as models one is naturally led to the positivistic conception which we have expounded above. Fresnel's undulatory optics and Maxwell's electro-magnetic theory are equally true according to Poincaré (op. cit., Ch. X) in so far as they lend themselves to the same differential equations. These "teach us, after Maxwell just as before him, that there is a certain relation between something and some

[1] *Électricité et optique*, 1901; *La science et l'hypothèse* (1902), ch. XII, translated into English by G. B. Halsted as *Science and Hypothesis* (New York, 1905).

other thing; only this something was called formerly *motion,* now it is called *electric current.* But these designations were nothing but images substituted for the real objects which nature will hide from us for ever. The only reality we can reach is the true relations among real objects, provided the same relations obtain among objects as among the images by which we are compelled to replace them. As long as we know these relations the fact that we find it convenient to substitute one image for another matters little" (p. 190).

Nearly related to this order of ideas is the criticism of scientific theories made by us in our *Problems of Science.* It is true, that a clearer positivistic criterion prevents us from using certain phrases, like those mentioned above, in which allusion is made to an unknowable nature of things (a notion to which no sense can be indeed ascribed). After rigorously defining the *positive content* of theories as the sum total of verifiable facts which they enable us to predict, we introduce what we call *representative hypotheses.* By this we mean those hypotheses which refer to the system of images upon which theories are built and which can be partly *indifferent* to the predicted facts. But in contradistinction to the more or less explicit tendencies of some of the preceding thinkers, we vindicate the value of the representative elements of knowledge. For according to us there is no science made which does not necessarily find an extension in a science to-be-made. The hypotheses which are indifferent in the limited sphere of the actual theories acquire significance from the point of their possible *extension.*

In any case these hypotheses express the need for *extending* the representation of certain phenomena be-

yond the temporary limits which seem to interrupt the *continuity* and *unity* of nature. When a body is decomposed and disappears in a solvent, we still believe it to be preserved by imagining molecular particles, and when the strokes of the hammer cease and the vibrations of the anvil gradually die away, we still pursue the motion under the form of heat. The importance of representative hypotheses is to be found thus not only in the *logical consequences* which can be deduced from them but also in the auxiliary hypotheses which they are capable of suggesting and which express in a certain sense their *psychological consequences*. A complete consideration of scientific theories shows that they are "systems of hypotheses on the one hand and systems of actual or possible experimental verifications on the other hand." But this offers us, on a higher plane, a counterpart of what has been revealed to us by the analysis of brute facts, which consist of two invariably conjoined elements, of voluntary acts and the sensations depending on them. By means of theories we get hold of "scientific facts" owing to systems of hypotheses which form the voluntary premises of verification and which are susceptible of assuming various representative forms. Just as our belief in the reality of a simple object is connected with so many forms of expectations which the imagination colors in different manners, so the understanding of a scientific fact is the richer the greater the variety of the images which we can see behind the possible experiences.

The true value of the abstract conception of science thus consists not in the ability of depriving the positive content of all images but in the possibility of rediscovering the same content in several systems of images. When

well chosen each of these systems proves to be an addition
to the economy of knowledge, making new predictions
depend on old habits of thought.

40.   PRAGMATISM

These considerations present an obvious analogy to the
concept of the logical structure of theories expounded in
the preceding chapter as well as to the philosophy of
pragmatism connected with it.

The fundamental concepts of a deductive theory have
been shown to be implicitly defined, from the point of
view of logical order, by their relations. Some of these
concepts are assumed at the beginning as postulates,
others are progressively deduced form these in the form
of theorems. "To deduce" means thus to develop the
meaning of the concepts contained in the principles. If we
now select from the consequences at which we arrive in
this way particularly those which denote facts of possible
experience, we shall find the *positive meaning* of the theory
independently of the images that form its representative
aspect. But just as in the case of the formal consideration
of a logical system where we meet with Plücker-Hesse's
principle of substitution and transference, so in the pres-
ent instance the true value of the abstract conception of
the theory is to be found not in an empty abstraction but
in the multiplicity of intuitions which are presented to us
by the various perspectives of facts.

It is exactly through his studies of mathematical logic
that Charles S. Peirce was led to define that method of
analysis which—according to the principles used by
Berkeley for explaining the concepts of matter, substance,

etc.—is capable of making our ideas clear.[1] He expresses his thought in the following formula: "The meaning of a theory (idea, etc.) consists in the *practical consequences* which can be deduced from it." The practical consequences are simply the possible experiences which bestow objective value upon knowledge, if knowledge is regarded from the point of view of subjective interest as "rules for action." But the formula was to become famous through the interpretation given to it by William James.[2] The recognition that a true belief implies an appeal to our functional possibilities is interpreted by him to mean that "the true is only the expedient in the way of our thinking just as 'the right' is only the expedient in the way of our behaving." [3] Instead of being regarded objectively (as the invariant in the order in which sensations follow upon volitions) the true appears as a variety of the useful. It is no longer conceived as a norm independent of desire and fear but as an instrument of success belonging to the same category as the *will to believe*.[4]

Pragmatism is conceived by James as the first step towards a *radical empiricism* which is to culminate in a *pluralistic view* of the universe. The utilitarian justification of science, the reserved evaluation of its critics (let alone the fragmentary conception of reality which empiricism never transcends) reduce themselves thus to a depreciation of science. With an insidiousness which is

[1] "How to make our ideas clear" (*Popular Science Monthly*, 1878).

[2] The most complete expositions of his thought are to be found in the following works: *Pragmatism* (New York, 1907); *The Meaning of Truth* (London, 1909); *A Pluralistic Universe* (London, 1909).

[3] *Pragmatism*, p. 222.

[4] Cf. James, *The Will to Believe and Other Essays in Popular Philosophy* (New York, 1899).

the more dangerous the greater the abilities of the author, James has succeeded in paving for thought the convenient road of "pragmatic breadth," offering us on the whole a *philosophy of water-tight compartments*.

Pragmatism has wide ramifications in numerous similar philosophical doctrines exhibited by the intuitionism of Bergson or the recent varieties of a romantic-utilitarian idealism. The rapid success of this movement can be accounted for by the fact that it echoes the complex motives of the contemporary soul torn by incurable discords. All these doctrines agree in denying the theoretical significance of science (which reduces itself for them to a set of practical prescriptions); they try to oppose to it other modes of knowledge or aspects of reality, such as artistic or religious vision.

We are not concerned in the present work with these developments. We already have had an occasion to expound elsewhere our views of pragmatism [1] as well as our conception of the fundamental *identity* which we find between *science and religion*. We achieve this by basing our construction of real invariants upon an affective activity which is of a religious nature.[2] A more thorough examination of science regarded (from a wider point of view than that of Mach) in its relations to rationalism would confirm and clarify these views in more than one sense.

These considerations imply an integration of the economic and utilitarian conception of science, without

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[1] Cf. *Scienza e razionalismo* (1912), ch. I. The points of similarity and difference between our view and pragmatism which have appeared already in the *Problems of Science* are sharply brought out by Josiah Royce in his preface to the English translation of this work (Chicago, 1914).

[2] *Scienza e razionalismo,* ch. IV.

detracting in the least from its *biological significance*.
We have only to recognize that the biological interest
shows itself not only in the saving of vital energies but
also and especially in their preservation and hence in
the exaltation of the fundamental feeling of life which,
in our opinion, finds its expression in religious activity.

41. The Logic of Systems

From all our preceding considerations we shall try
now to deduce as a conclusion the most general logical
views which serve for scientists both as methods of dis-
covery and of proof.

We have already pointed out the limitations of induc-
tion. It is never capable of yielding results and general
principles whose certainty is so established that they
stand in no need of verification. We have also shown that
it is equally impossible to verify the consequences of an
hypothesis or a principle independently of the acceptance
of other hypotheses (§ 36). Inductive logic thus gives
way to a logic of system or theories, where we deal with
the construction, demonstration, and evolution of sys-
tems of concepts with which we try to represent various
orders of phenomena.

The scheme of the process of inference presented by a
theory has been described by us as consisting of four
stages:

1) Observations and preliminary experiments;

2) Concepts or a system of concepts which represent
it hypothetically;

3) Deduction;

4) Verification.

When the verification turns out to be successful for a *certain order* of experiments, the scientific procedure used appears to us as *closed,* and the hypotheses expressed in the theory are regarded on the whole as proven. Of course we are still left with the task of establishing—by means of a proper criticism—their real *positive content* and of separating those elements of the representation that can be regarded as indifferent from this point of view.

But the proof reached in this way can be regarded only as provisional and approximate. This is at least the case when the *theory* is interpreted in a *concrete sense* implying a certain *phenomenalistic determinism* that is capable of yielding positive predictions and not only limitations of expectations.

Indeed, if it were possible to deduce continually from a scientific theory new consequences admitting of a positive verification whose degree of approximation is higher than any assignable limit, such a theory would possess an infinite extension evidently transcending the possibilities of science. The hypothesis of an unlimited agreement with experience can be assumed only when it is a question of *theories* interpreted in an *abstract sense,* of theories regarded as systems of principles which have to be completed in every application by means of additional hypotheses. But in such a case it is impossible to tell whether the verification, whatever its result may be, refers to the theory conceived in this way or to its complementary hypotheses.

Apart from this sense, which we shall consider later, the impossibility of a theory with an unlimited extension follows a priori from the fact that the *concepts* of

which it is made up are *abstract constructions* which cannot embrace all reality.

We thus can regard it as a general principle that *to pursue the deductive development of theories sufficiently far means to arrive at negative results* which mark a limit to their validity. If the negative and well controlled result cannot be easily explained by means of a simple correction through secondary hypotheses (which are compatible with other experimentally established consequences), this experiment, even if it is a single one, signifies the death of the theory. We usually experience in such a case that feeling of sadness which accompanies the end of all living things.

But is this sadness justified? "The physicist ought to be on the contrary very joyful," we hear Poincaré say,[1] "when he has to give up one of his hypotheses, for in this way he finds an unexpected occasion for discovery. The absence of verification means that we are confronted here with something unforeseen and extraordinary, with something suggesting the possibility of discovering new and unknown things."

But the physicist finds himself in an unpleasant predicament. It is impossible for him to retain the framework which fitted his old observations and experiments and at the same time he is unable to build another one capable of accommodating the new experiment. Instead of counting with what he has already accomplished he has to begin his work anew. Under such conditions his sadness is in reality all too human.

Whatever the case may be, the *life of a theory* is enclosed within the cycle described by us. It proceeds

[1] *Science et hypothèse* (p. 178).

from a limited to a more extended experience, whence it rises to a new conceptual construction which possesses a higher degree of generality and accuracy.

Now this inductive phase of science (which reappears in new forms in the transition from system to system) does not lend itself to precise methodical rules. All we can say about it is that *deduction* reveals itself here as the *proper instrument of generalization*. The most general principles of the new theory are nothing but theorems of the old deduced in an irreversible manner from the premises, to which certain corrective hypotheses are often added. For instance, the mechanistic hypothesis involves as a consequence the principle of conservation of energy which—owing to its greater generality—proves to be capable of explaining also facts agreeing with the mechanistic hypothesis.

But the *true generalization* of a scientific theory consists not only in the logical fact of the adoption of a more general system of hypotheses but also in the *extension* of the theory itself to new *orders of phenomena*. It is noteworthy that this extension can often by itself give rise to a *correction* of our original hypotheses by anticipating the test of the negative experiment. This is clearly seen in the classical example of the Newtonian gravitation. From Kepler's laws of planetary motion we deduce by calculation the attractive force exerted by the sun upon the planets and then the action of the planets upon their satellites and especially that of the earth upon the moon. A comparison of the latter force with that acting upon bodies on the surface of the earth leads to the brilliant idea of a "mutual attraction between any masses whatsoever." But when Newton's theory is gen-

eralized in this way, it necessarily implies a correction of Kepler's laws, which remain valid only as a first approximation (correction of the third law, perturbations of planetary motions).

Classical mechanics seems to have been invalidated also by the experiments upon cathode rays which led scientists to doubt the postulate of the constancy of mass for great velocities. But this negative result could be predicted from the moment when mechanics became so extended as to embrace electro-magnetic phenomena. The experiments of Kaufmann are as a matter of fact in full agreement with the calculations of Abraham.

On the other hand, the purpose of the negative test of an experiment is not always to reveal things that were to be expected from the known facts. Its function is also to correct the errors involved in certain conceptual constructions and to invite us to search for what an appropriate criticism would have suggested beforehand as more probable and conformable to sufficient reason.

This is true of the experiments of Michelson and Morley which turned out contrary to the original purpose of offering a proof for the absolute motion of bodies with respect to the ether. The experiment does not furnish us here with a new motive but only with an occasion for doing away with a fetish and for accepting the critical conception of the relativity of motion, which enables us to satisfy the rational demand for explaining the motions of bodies without introducing invisible systems of reference.

42. A Comparison with the Hegelian Dialectic

Whatever part experience may play in the evolution of a given scientific system, this evolution presents, from the point of view of the progressive development of concepts, a notable similarity to the idealistic doctrine and especially to Hegel's dialectical scheme:

thesis, antithesis and synthesis.

We indeed can say that in the logic of scientific systems the inductive development of a concept follows upon the negation or the contradiction of the thesis, thus transcending the original position.

Certain fundamental differences have, however, to be pointed out:

1) The development of the concept which leads in the Hegelian dialectic from affirmation to negation is prompted exclusively by an inner logical motive and not by the stimulus of or clash with external experience.[1]

2) The reaffirmation of the thesis in the synthesis is conceived—owing to the formal symmetry with the first transition from the thesis to the antithesis—as a negation of negation. In the development of a scientific system, on the other hand, the inductive phase that follows upon the negation of the theory is radically different from the deductive phase, since unlike the latter it is not subordinate to determinate logical rules.

3) The Hegelian dialectic describes not so much the

[1] If the Hegelian philosopher regards nature according to a poetic myth which sees in it the work performed by Spirit in the past, as something of which we have lost consciousness, he has to admit that this nature presents itself to our consciousness as something given; experience thus has to become for him as important as the study of historical reality.

evolution of logically defined concepts as movements of ideas—upon an affective basis—tending to embrace moral or social reality or to represent religious and senti-mental aspirations. It expresses the historical experiences of the romantic period, translating them into what Josiah Royce so well designates as "the logic of passions." [1]

"Great injustice is done to Hegel," [2] James says, "by treating him primarily as a reasoner. He is in reality a naïvely observant man, only beset with a perverse pref-erence for the use of technical and logical jargon. He plants himself in the empirical flux of things and gets an impression of what happens. His mind is in very truth *impressionistic*." And a little further (p. 92) he adds this malicious remark: "The only thing that is certain is that whatever you may say of his procedure, some one will accuse you of misunderstanding it." [3]

But how can the logic of passions ever have anything in common with the most developed logic of scientific systems?

The question requires, in our opinion, a rather thor-ough explanation. We believe that this is to be found in the view according to which the movement of science reproduces in a higher degree the forms of the historical evolution of all spiritual movements which can be best seen in the short life of ideas prompted by feelings. Here too concepts are formed by means of association of images, but intellectual inhibition does not succeed in checking them with rigid abstractions.

It is exactly for this reason that Hegel (who absolutely

[1] *The Spirit of Modern Philosophy* (Boston and New York, 1893).

[2] *A Pluralistic Universe*, p. 87.

[3] Cf. our essay "La metafisica di Hegel" in *Scienza e razionalismo,* III, 2.

lacks this inhibitory logical quality) is so antipathetic to the barbaric use made by the sciences of the word "concept" (Begriff) understood as a collective representation of an ill-defined class of objects. For he sees a distortion and dismemberment of reality in the abstraction which inhibits the free movement of the idea by separating it from the associations and relations with the images which it would naturally carry along.

We cannot enter here upon an examination of this criticism,[1] which was intended by Hegel as a means for distinguishing between what is "common" and what is truly general or "universal." Suffice it to say that the Hegelian dialectic simply expresses the *antithesis* of the intellectualistic view according to which systems of concepts are regarded *only* as immobile abstractions. The logic of scientific systems offers a *synthesis* of the two opposed conceptions. The true and concrete universal reduces itself for this logic to a progressive aspiration for an indefinite extension of the common and limited representations of sense objects. *Abstraction* is necessarily a *part* of such a process.

However paradoxical it may appear, we feel, nevertheless, justified in maintaining that the attempt to grasp the "concrete" outside all abstractions is itself an absolutely abstract abstraction and therefore empty and absurd. Human mind projects into this specter the unattainable limit of an infinite series of acts of thought each of which implies a certain degree of abstraction and an apprehension of a subordinate reality.

[1] See in general *Wissenschaft der Logik* (Werke, vol. III, Berlin, 1833) and especially ch. 5, "Die Lehre von Begriff," *Enzyklplopaedie*, part I, *Logik* (Werke, vol. III, pp. 320-21).

All degrees of reality indeed are, as we have seen, progressive constructions on the part of thought, which apprehends always by means of concepts. A concept is at the same time an abstract construction separating certain possible sense data from those which our activity actually claims as its own, and also an *act of unification* or an associative link which thought itself establishes between the data of the reality under consideration. Classical logic was concerned only with the first aspect, Hegel with the second. The true nature and function of concepts consists therefore for him not in the *separation* but in the *unification* of a class of elements which gives rise to the dialectical movement. But the great significance acquired by dialectic in the *evolution of systems* is due only to the fact that it is based upon that intellectualistic deduction which has for its presupposition the immobility of the abstract.

43.   A Priori Judgments and Conventions: Non-Euclidean Geometry

Let us examine now a little closer that critical phase in the evolution of theories in which mind rises from the rejection of hypotheses to a system of hypotheses and concepts that are adapted to a larger reality.

Are there any principles which in a certain sense determine or guide thought in this difficult enterprise?

If the question is restricted to the development of limited scientific theories, it may be answered that the theory which has been disproved by a negative experiment finds a criterion outside itself in other and more general theories with which it has to agree and from which it can

derive new principles. But at this point there arises the question whether the sciences exhibit a hierarchical order so that certain more general and abstract sciences possess a regulative value which is independent of the subsequent development of experience and whether they do not restrict, even though they may not determine, a priori the evolution of concepts and hypotheses.

An affirmative answer to this question has been given by Kant for whom knowledge is based upon *synthetic a priori judgments which necessarily determine the interpretation of every possible experience.*

The Kantian doctrine of the a priori has been invalidated by the development of science, especially by the construction of non-Euclidean geometries.

It was the belief of the first interpreters of this geometry (Gauss, Lobatchevsky, and among the Predecessors —Taurinus) that the observation either of geodesic or astronomical triangles could establish a difference between the sum of the angles of a triangle and two right angles. This would prove that *physical space* is different from that of the Euclidean intuition and that it corresponds to a value of the Gaussian curvature K which is small but different from zero. It has been, however, impossible to discover a real difference surpassing the errors of observation.

It is true, the Kantians may still maintain their views, despite this attempt at bestowing immediate physical validity upon non-Euclidean geometry. They may, for instance, interpret the eventual observations and experiments which would give rise to $K \gtrless 0$ not as a refutation of the Euclidean hypothesis but as results following from certain properties of the solid bodies which serve us as

instruments of measurement. But even if it were possible to maintain in this way a view akin to the Kantian, we should at least have to give up the proper thesis of Kant, according to which the possibility of experience is necessarily based upon the synthetic judgments of the Euclidean Geometry without which experience itself could not be interpreted. For we can, on the contrary, assign such a small value to K that all our experiences —within the limits of observable errors—may also agree with the hypothesis of a non-Euclidean space of the curvature K.

The Kantian thesis modified in the sense suggested above (which is the more plausible the smaller the values of K) has been advocated in recent times in a new form by Henri Poincaré. In his opinion the question of Euclidean geometry can have no experimental meaning, since no experiment refers *to space* or to *relations between* bodies and *space* but only to the *relations of bodies among themselves*. The axioms of Euclidean geometry are consequently simple *conventions* or covert definitions which serve as premises for physical hypotheses.

"If this is the case," he exclaims,[1] "what shall we then think of this question: Is the Euclidean geometry true?

"The question is meaningless.

"We may just as well ask whether the metric system is true and the old systems of measurement are false; whether Cartesian coördinates are true and polar coordinates are false. One geometry cannot be truer than another; it can only be *more convenient*. Now, the Euclid-

[1] *Science et hypothèse,* p. 67.

ean geometry is and will remain the most con-
venient. . . ."

This conception, which regards principles as conven-
tions and which the author has amply developed (and
perhaps modified a little in his later works), does not
prevent him from recognizing the indispensable part
played by experience in the genesis of geometry. What he
denies is the supposition that geometry can be, even
partly, an experimental science:

"If geometry were an experimental science, it would
have only an approximate and provisional validity. And
what a crude approximation!

"Geometry would then be the study of the motions of
solid bodies. But in reality it does not deal with natural
bodies at all; it has for its object certain ideal solid
constructions which are absolutely invariable and which
are nothing but a simplified and remote image of natural
bodies.

"The notion of these ideal bodies is derived from our
mind, and experience plays only the part of an occasion
stimulating us to call them forth."

The object of geometry is the study of a particular
"group." But the general concept of group pre-exists in
our mind, at least potentially. It imposes itself upon us
not as a form of our sensibility but as a "form of our
understanding." The function of experience is thus to find
among these groups the most convenient *standard*.

It is clear that we are here in the presence of a renewed
form of Kant's Esthetics, in which "the arbitrary" has
taken the place of the "a priori."

We have amply discussed and criticized this thesis in
Ch. IV of the *Problems of Science*. It is neither space nor

the relations of bodies to space that form the object of geometry, but a certain class of relations (spatial) among bodies. These relations are of course a simple abstraction of more complex physical relations. Understood in a concrete sense, geometry finds its continuation in kinematics, mechanics, and optics. For the so-called geometrical experiments are in reality nothing but experiments of measurement which involve the properties of matter and light and in which we disregard certain elements that seem to be *statistically* negligible.

If we admit that geometry is from a concrete point of view only a part of physics, its axioms will be seen to figure along with other hypotheses in the scientific system which we construct as a representation of reality. What does it mean then to choose certain hypotheses from these in order to erect them *by convention* into rigorous principles?

It seems to me that this is only an unhappy expression for designating the arbitrariness involved in every supposition as an act of will governing experience. If f, φ, ψ . . . are equations established among a certain number of unknown terms, I can demand first that f should be satisfied—a fact restricting the arbitrariness in the variation of the unknown terms—and can examine then whether $\varphi \psi$ . . . are compatible. But if the answer is negative, I must not forget that the incompatibility brought here to light belongs not to the partial system φ , ψ . . . but to the complete system f, φ, ψ . . . in which f, despite the convention used, plays the same part as the other equations.

The hypotheses which Poincaré erects into principles accordingly remain, after his convention, and exactly

because of it, only hypotheses. They are in reality the first hypotheses which the author uses in order to restrict the meaning of the subsequent ones and consequently the content of the expectations which they may imply. We must not be deceived by the declaration that the hypotheses erected into principles assume a *rigorous* value by convention. This declaration adds nothing to the simple act of making a supposition. Indeed every supposition is always understood and treated in the theoretical development of science as a logical supposition and consequently as rigorous. Its approximate character appears only in the verification.

The refutation of Poincaré's conventionalism thus brings us back (perhaps with a more precise notion of the physical value of geometrical hypotheses) to the concrete conception of geometry as it was entertained by Riemann, Helmholz, Clifford, etc. But this conception is acquiring an exceptional importance at present owing to the new theories of Albert Einstein. Its relation to the latter is brilliantly illustrated by Eddington in the introductory dialogue to his well known work *Space, Time and Gravitation*.

44. THE RELATIVISTIC ELECTRO-MAGNETIC PHYSICS AND THE RATIONAL REQUIREMENTS OF KNOWLEDGE

The development of the ideas which led through the recent crisis of mechanics to Einstein's doctrine offers a means for refuting the conception of a hierarchical order of scientific concepts and thus for throwing new light upon the question of the a priori.

The significance of this crisis is to be found in the following two conceptions of the structure of nature which were present in the theories of physics:

1) According to one, the most familiar intuitions of the phenomena of motion were assumed in the concepts and principles of the Newtonian mechanics in order to explain the universality of physical facts. This meant the postulation of an absolute space (or motion), absolute time, Galilean inertia, and of action at a distance;

2) According to the other (which tried hard to agree with the first), electro-magnetic phenomena were comprised in the scheme of a theory which Faraday, Maxwell, and Lorentz based upon the contiguity of the causal relation, that is, upon the continuous propagation of action both in space and time.

The two doctrines turned out to be irreconcilable. It also became apparent that the extended field of optic, electro-magnetic, and radio-active phenomena could hardly be reduced to mechanical explanations, and that, on the contrary, it was possible to obtain a convenient electro-magnetic explanation for mechanical phenomena.

The successive stages of this progress of ideas were marked not only by experiments which stimulated the development of the electro-magnetic doctrine but also by a criticism of concepts owing to which physicists acquired a clearer consciousness of the rational demands of knowledge.

This statement, in so far as it is contrary to the commonly accepted opinion, may appear as paradoxical. If we are to regard absolute time and space as incontestable demands of reason and if, on the other hand, we

accept the Newtonian compromise of action at a distance,[1] the rejection of the Newtonian concepts will be due only to the compulsion of experiments. The story of the Michelson and Morley experiments, to which the first place is assigned in the exposition of the doctrine of relativity, seems to offer a confirmation of this view. But how can it be maintained that absolute motion expresses a rational demand?

On the contrary, the earliest rationalistic philosopher, Parmenides of Elea, arrived by the sole force of criticism at the notion of the relativity of motion.[2] Rationalists like Kepler and Descartes followed him in this direction. The relativistic thesis is an unavoidable consequence of the principle of sufficient reason, if motion is defined and explained uniquely in relation to matter as we see it moving. But the Parmenidean thesis had been interpreted as implying a denial of the *concept* of *motion* without which a science of mechanics seems to be impossible. It is in this sense that Democritus and later Galilei found it necessary to postulate a space of absolute reference. Newton espoused this postulate because he was under the illusion that our intellectual habit of regarding bodily motion as something definite in itself, forms a rational demand and because he found, on the other hand, an experimental confirmation for it in the property of permanent axes of rotation (gyroscope). Modern critical thought, to be sure, had already turned to the relativistic

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[1] Newton himself saw in this only a provisional working hypothesis; he wanted to explain attraction (after the manner of the Democritean hypotheses developed by Lesage and Fatio de Dullers) by reducing it to pressures and impacts. Cf. *Optica*, Quaestio 313, op. cit., p. 153.

[2] Cf. Enriques, "La relatività del movimento nell' antica grecia" (*Periodico di Matematiche,* 1912).

view at the time when the hypothesis of an immobile ether seemed still to offer a necessary basis for electro-magnetism. Michelson as a matter of fact undertook his experiments in the naïve hope of finding evidence for the motion of bodies with respect to the ether. Suffice it to say here that Mach had looked for an explanation of the above mentioned property of permanent axes in the influence of stellar masses.[1]

As to the relativity of time we have to admit that the Einsteinian criticism (which was suggested by Lorentz' hypothesis of *local time*) appears at the first blush as new and paradoxical. This criticism follows necessarily from the principle of relativity only when we accept the experimental datum of the constancy of light. But if this datum makes us subscribe to the relative rather than to the absolute character of motion it is because this choice does not involve anything that clashes with the ways in which our understanding functions. This is at least true (as the case actually is) as long as the indifference in regard to the order of simultaneity refers only to phenomena which (we of course take here into account the insuperable limit set by the velocity of light to the propagation of actions) cannot influence one another physically; in this way our ideas of causality are not affected.

Now, the Einsteinian doctrine signifies the triumph of

[1] We may also refer to the analysis given by ourself in the *Problems of Science* (ch. V), where we found the statics and dynamics of incipient motion upon any system of reference whatsoever. We may add that this construction can naturally be extended, leading to a complete system of relativistic mechanics (dispensing with electro-magnetic theories). We have only to substitute the law of inertia by Einstein's postulate, according to which the laws of dynamics must have a co-variant character with respect to different systems of reference (remark of Levi-Civita).

a more satisfactory insight into the nature of the causal relation. The principle of contiguity of action carries the day over action at a distance. It was only experience which made us accept this hypothesis for a time. But despite the criticism of positivists [1] it was in reality never able to satisfy human reason, except perhaps in the case of a few Kantian zealots.

The lesson imparted by this marvelous scientific achievement does not militate against a well-understood rationalism but only against a philosophy which—under the pretext of rational demands—consecrates solely the mental habits and illusions involved in it.

Judgments a priori found their justification for Kant in the conception that the activity of mind which introduces order into sense data must assume a special form imposing itself necessarily upon all knowledge. This necessary form he found not only in the principles of logic but also in the intuitions of Euclidean geometry and Newtonian mechanics, from which he derived once and for ever the regulative concepts and axioms of science.

But this conception of the constructive activity of mind as it manifests itself in knowledge is limited and poor. It is certainly absurd to separate the form from the

[1] After showing that Newton himself wanted to explain gravitation by reducing it to pressures and impacts Stallo (op. cit., ch. V) avers that Huygens—despite this—regarded the principle of attraction as absurd and that according to Leibniz it meant to appeal to an incorporeal and inexplicable power. J. Bernouilli too denounced this supposition at the Paris academy as "shocking to minds accustomed to accept only incontestable and evident principles in physics." Stallo comes finally to the melancholy conclusion that even to-day physicists do not follow John Stuart Mill in accepting action at a distance; they still find it impossible to give up the idea of reducing forces to actions transmitted by contact.

content in the work of a poet; every new content has a corresponding new form with which the artist is supplied by his imagination. Similarly, and to a greater extent, in the evolution of science the inexhaustible activity of mind constructs new forms in order to embrace a wider order of facts and to express a richer consciousness of its own laws.

The demands of reason thus present themselves not as given once and for ever in relation to an empty form but *progressively as functions of experience*. It is as interpretative criteria of experience that they assert themselves. And if there is anything constant in them it is only certain aspirations and tendencies but not rigid principles.

45. The Hierarchical Order of the Sciences and the Unity of Knowledge

Among the profound aspirations which the human mind tries to satisfy in the progress of knowledge two are especially noteworthy:

1) The tendency aiming to grasp reality as a *unity;*

2) The tendency which finds its expression in the general form of sufficient reason.

Our preceding description serves to refute not only the specifically Kantian doctrine of synthetic a priori judgments but also the conception of a hierarchical order of the sciences, as we find—in a different epistemological form—in Auguste Comte's classification of the sciences.

We have examined elsewhere the significance of this classification, which projects into reality a certain historical and psychological order of human abstractions. We have also shown how it is connected with the meta-

physics of materialism and how it sanctions a particularistic conception of science.[1]

From this we infer that reality is only a tissue of independent causal series. In this way we can understand how contingency finds a place beside a mitigated determinism in the philosophy of a John Stuart Mill or a Karl Pearson. Finally the development of this particularism leads to the pluralism of William James.[2]

Empiricists and pragmatists try hard to confine scientific theories to the narrow sphere of practical action, at the same time depreciating and appreciating their value in the name of the useful, and absolving them from their mutual incompatibilities. But the struggle does not subside among the concepts which have been discarded as useless. On the contrary, it continually breaks out with greater force in the work of scientific thinkers. It is a struggle for the unification of *knowledge*.

The very notion of the hierarchy of the sciences already implied, in a certain sense, that they form a unity. But the inter-connectedness of knowledge is not stressed here sufficiently, for general knowledge is regarded as anterior to and independent of particular knowledge. Since the phenomena are so to speak assigned to different spheres among which we assume a priori certain relations, the result is that a one-sided pressure is exerted upon scientific work. In its attempt to keep up with the specializa-

[1] "Il particolarismo e la classificatione delle scienze" in *Scienza e razionalismo* (ch. V).

[2] The subsequent development of idealistic systems of philosophy bears witness, in our opinion, only to the demand for unity, without, however, satisfying it. A unification of thought can certainly not be found in a philosophy which arbitrarily separates the fields of our mental activities, and which uses convenient pretexts for going so far as to refuse to take cognizance of the progress of science.

tion of instruments and methods this work tends to favor certain directions of study to the detriment of certain others.

The true *unity* of knowledge does not deny the limits of practical possibility which involve a division of labor. But it has also as one of its conditions that no *theoretical* limit should be set to the *freedom* of research, that no philosophical value should, for instance, be attributed to the contingent and provisional distinctions obtaining among the different branches of knowledge. In this way it is not the technical means that have to indicate the directions in which problems have to be studied. On the contrary, every important scientific problem has to determine (from time to time) special coördinations of technical means and corresponding organizations of scientific workers.

These views have been advocated by us, not without opposition, in the social field of scientific institutions and the organization of scientific studies. We have not lost the hope that they contain fruitful germs. This offers us a certain compensation for the bitterness entailed by every struggle for ideals.

But let us come back to the philosophic question. Every aspect of reality, every act of thought with which we try to get hold of a certain order of phenomena tends to expand into a system virtually aiming to grasp the whole universe. It is this unlimited extension that gives rise to the struggle among systems as well as to their progressive unification.

The true significance of the rational demand which reveals itself in this progress does not consist for us in the assertion of a particular representation and hence of

a natural order of knowledge. On the contrary, we regard as legitimate the different perspectives of reality which translate the same positive content into different representations. In the last analysis the unity of knowledge postulates the possibility of extending beyond all limits the series of real invariants. This possibility expresses the condition of the unity of the human mind manifested in the free development of ideas.

This is the highest postulate of science. Disregarding romantic intoxications, it posits an *intelligible order of nature,* not as a pure external datum or as a fixed goal to be attained, but as an infinite progress through whose stages our reason passes.

46. THE PRINCIPLE OF SUFFICIENT REASON

In the struggle among scientific systems, and especially in the critical phase of their evolution, it often occurs that a certain order of concepts proves inadequate for an extended experience, and our mind tries then to use new conceptual frameworks for the new and the old data. It is then that we become confronted with the supreme demand which finds its expression in the Leibnizian principle of sufficient reason.

We have already examined this principle elsewhere.[1] Although we must to-day take a wider view of its historical development, we have nothing to change in the conclusion reached by us as to its significance. The principle of sufficient reason does not yield any axioms from which knowledge can be deduced a priori (as so many claimed, for instance, in the case of the law of inertia, and up to

[1] *Scienza e razionalismo* (II, 3).

the attempt—just as famous as it was disastrous—of Schopenhauer). It only presents *conditions which have to be satisfied in order that a certain conceptual representation may fit a certain order of data.*

In this sense it can be said that it offers a logical criterion for the evolution of scientific theories. Let us suppose that a new datum does not fit the system in accordance with which we wish to interpret it. We have then to reject the new concepts which are constructed to embrace the more extended reality, if what we regard as capable of substitution in the physical realm does not appear as "equal" in the domain of representative concepts, or if the symmetry of the "causes" does not correspond to an analogous symmetry of the "effects," etc.

But two points are noteworthy here:

1) The criterion of sufficient reason has rather a negative than a positive function; [1] it namely limits the choice of scientific explanations;

2) In any case it is not to be translated into a rigid form but, as we have said above, into certain mental tendencies which fit reality in an extended sense.

The fact that these tendencies can be satisfied in dif-

[1] I shall cite only one example. According to Kant the principle of the conservation of matter in chemical reactions is an a priori judgment. Chemists, on the other hand, agree in regarding it as based upon the experiments of Lavoisier. There can be no doubt that the chemists are right here and that the attempts of Landolt and Heydweiler to find a more exact verification of it are far from meaningless. For reason can lead us to look in chemical transformations for something constant, but it cannot tell us that this "something" can be translated into certain units of measurement, for instance, of weight. It is noteworthy that according to most recent atomic doctrines (Rutherford, Bohr) the weight of a compound is not exactly equal to the sum of the components, especially in atomic transmutations. If these contain something constant, it is the number of atoms or the energy properly determined.

ferent manners and degrees has furnished positivistic criticism with a motive for reducing the content of the intuitions by which causal relations are accompanied in our thinking (that is, in the rational interpretation). This reduction is already to be seen in David Hume's analysis as we pass from the *Treatise of Human Nature* to the *Inquiry Concerning the Human Understanding*,[1] since in the latter work the condition of the contiguity of causal action is dropped. It is true, Hume had distinguished between two aspects of causality (as a *natural* and as a *philosophical* relation) and pointed out the bond connecting the idea of the cause with that of the effect. But his interpreters (for instance, Stallo, cited in 37) mention the view asserting that the order of the genesis of concepts is identical with the order of the genesis of things, only in order to denounce it as futile. We may condemn the ancient realistic view which objectifies rational relations as entities outside our mind. But this does not mean that we have to deny the postulate of the possibility of knowledge according to which we try to represent—as far as it is possible—conditions of real sequence by relations among concepts. Even if this postulate does not possess the absolute significance which Kant ascribed to it, its importance is capable of manifesting itself in the subjective satisfaction of scientific thought as well as in the ever-growing success which science can achieve as an instrument of prediction. A manual instrument is the more productive, the more it is adapted not only to the material to which it is applied but also to the hand which uses it.

[1] Green and Grose's edition, London, 1898. Cf. especially *Treatise*, vol. I, pp. 463-64; *Essay*, vol. II, p. 63.

These views have already been expounded by us in Ch. III of our *Problems of Science*.

Our analysis of cause throws light especially upon the difference of the psychological significance existing between the explanations in which we show "how" phenomena take place and those in which we try to account for the "why." It is easy to recognize the points of agreement and disagreement which can be found between this analysis and other views dominated exclusively by logico-mathematical criteria (somewhat arid to be sure), as we see, for instance, in the recent criticism of Bertrand Russell.[1]

Without detracting anything from Mach's conception of the interdependence of phenomena, we believe that the idea of cause does not lose anything of its value because it has to be regarded as a conceptual limit or, if one prefers, as a rational demand of the possibility of knowledge. We have already said why such rational demands—although their significance is relative—are not to be regarded as idols which have to be destroyed. Even if they betray habits of thought which try to explain the more known by the less known leading to more familiar phenomena, they always express a legitimate aspiration. But it is not to be assumed that *all* rational demands are to be reduced to habits which can be superseded. For whatever part we may assign to the environment as an epigenetic factor in the development of the organs of thought, there remains a certain limit which is preformed in the structure of these organs and in the physical laws which govern their function.

[1] "On the Notion of Cause," *Mysticism and Logic* (London, 1919). Cf. G. Scorza, *Periodico di Matematiche*, January, 1921.

Without entering upon an examination of this controversial biological problem we may in any case conclude that our scientific representations have to bear the seal of our mental activity. We may repeat that the progress of science consists exactly in a greater harmony between thought and reality, and consequently in a double adaptation: of thought to the data of experience, and of experience to the forms of thought.

A critical examination of the recent achievements of science will show in a more proper light the significance of the conditions involved in scientific development. We have in mind such factors as the search for more and more comprehensive invariants, the demand for accounting for any agreement, even of a purely numerical nature,[1] a more complete and more precise univocal determinism (which conditions the principles of symmetry and of maxima and minima in mathematical physics), the philosophy of critical realism [2] asserting the indifference of natural laws with respect to any system of reference and hence the covariant character of laws in regard to different systems. While logicians impoverish the old concepts of cause with their analysis, the genius of Einstein discovers in it a more plastic and also a richer content.

[1] We have only to think of the place occupied in Einstein's theory by the equivalence of inertial and gravitational mass.

[2] It would be good for certain idealistic interpreters to realize that the philosophical significance of Einstein's theory is to be found here and not in idealism.

APPENDIX

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